THE ELEMENTS

OF

HYDROSTATICS

AND

HYDRODYNAMICS.

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ELEMENTS OF HYDROSTATICS.

SECTION I.

GENERAL PROPERTIES OF FLUIDS.

Art. 1. A fluid is a body which can be divided in any direction, and whose parts can be moved among one another by any assignable force.

Elastic fluids are those whose dimensions are increased or diminished when the pressure upon them is diminished or increased. Non-elastic fluids are those whose dimensions are independent of the pressure.

Water, mercury, and probably all other liquids, are in a small degree compressible. Their resistance however to compression is so great, that the conclusions obtained on the supposition of their being incompressible, are in most cases free from any sensible error.

2. Let *DEF* (fig. 1.) be a piston without weight exactly fitting an orifice in the plane *ABC*, which forms the side of a vessel containing fluid. It is manifest that the fluid can make no effort to move the piston in any other direction than that of a normal to its surface, the piston may therefore be kept at rest by a force applied at some point *G* in it, and acting in a direction *HG* perpendicular to *DEF*. A force equal and opposite to this is called the pressure of the fluid on *DEF*.

3. The pressure of a fluid at a given point is measured by the quantity p, $p\kappa$ being the pressure of the fluid on an indefinitely small area κ contiguous to the given point.

When the pressure of a fluid on a given surface is the same, wherever that surface is placed, p is the pressure on an unit of surface. When the pressure on a given surface, varies with the situation of the surface, p is the pressure which would be exerted on an unit of surface, if the pressure at each part of the unit of surface were equal to the pressure at the given point.

4. Axiom. When a fluid is at rest, any portion of it may become solid without disturbing either its own equilibrium, or that of the surrounding fluid.

For as long as the fluid remains at rest, it makes no difference whether the parts of which it is composed, are moveable among one another, and capable of being divided in any direction, or not.

5. Fluids press equally in all directions.

Let $A\,b\,c$ (fig. 2.) be a very small prism of fluid in the interior of a fluid at rest; then (Art. 4.) the equilibrium of $A\,b\,c$ will not be disturbed, if we suppose it to become solid. Now if R be the accelerating force at A, $A\,b\,c$ is kept at rest by the pressure of the surrounding fluid on its ends and sides, together with R.(mass prism) acting in the direction of the force at A. But if the prism remain similar to itself while its magnitude is diminished indefinitely, R.(mass prism) vanishes compared with the pressure on either of its sides; (for the former is proportional to $A\,a^3$, the latter to $A\,a^2$;) and we may consider the prism to be kept at rest solely by the pressures on its ends and sides: and these pressures are respectively perpendicular and parallel to ABC, therefore they must be separately in equilibrium. And since the pressures on Ab, Ac, Cb are in equilibrium, and perpendicular to the sides AB, AC, CB of the triangle ABC, they are proportional to those sides; hence if $p\,.\,Ab$, $q\,.\,Ac$ be the pressures on Ab, Ac respectively, $p\,.\,Ab : q\,.\,Ac = AB : AC$, therefore $p = q$. But p, q measure the pressures of the fluid at A perpendicular to $A\,b$, $A\,c$ respectively, and $A\,b$, $A\,c$ may be taken perpendicular to any two given lines, therefore fluids press equally in all directions.

Cor. 1. Suppose the sides of the base of the prism to be indefinitely small compared with its length; then if the pressure on ABC be increased or diminished in any degree without disturbing the equilibrium of $A\,b\,c$, the pressure on $a\,b\,c$ must be equally increased or diminished. Hence if F, G, H...... M, N, P (fig. 3.) be any series of points in a fluid at rest, so taken that the straight lines FG, GH......MN, NP may be wholly within the fluid, and the pressure at F be increased or

diminished without disturbing the equilibrium of the fluid, the pressures at G, H......M, N, P will be equally increased or diminished.

COR. 2. If the fluid be acted on by no accelerating force, the pressures on ABC, abc must be equal; therefore pressure at F = pressure at G = ... = pressure at N = pressure at P: or, the pressure is the same at all points in a fluid at rest acted on by no accelerating force.

6. Let the forces P, Q, R, &c. be in equilibrium when applied to pistons A, B, C, &c. fitting cylindrical apertures in the sides of a vessel filled with fluid. Let a, b, c &c. be the areas of the pistons, and suppose the fluid to be acted on by no accelerating force. Then since the fluid is at rest, the pressures on an unit of the surface of each of the pistons must be equal,

$$\therefore \frac{P}{a} = \frac{Q}{b} = \frac{R}{c} = \&c.$$

7. Let the fluid be incompressible; p, q, r, &c. the distances of the pistons A, B, C, &c. from fixed points in the axes of the cylinders in which they play; $p + \delta p$, $q + \delta q$, $r + \delta r$, &c. their distances from the same points after they have been moved in any manner. Then since the volume of the fluid in the vessel remains the same,

$$a \,.\, \delta p + b \,.\, \delta q + c \,.\, \delta r + \ldots\ldots\ldots\ldots = 0.$$

Also since the forces are in equilibrium, $\frac{P}{a} = \frac{Q}{b} = \frac{R}{c} =$ &c.

$$\therefore P \,.\, \delta p + Q \,.\, \delta q + R \,.\, \delta r + \ldots\ldots\ldots\ldots = 0.$$

δp, δq, δr, &c. are the virtual velocities of the pistons A, B, C, &c. to which the forces P, Q, R, &c. are applied.

SECTION II.

ON THE EQUILIBRIUM OF NON-ELASTIC FLUIDS ACTED ON BY GRAVITY.

ART. 8. THE specific gravity of a body is the weight of an unit of its volume.

9. The density of a body is the quantity of matter in an unit of its volume.

10. Let W, M, V be the number of units of weight, mass, and volume contained in the weight, mass, and volume of a given body, S its specific gravity, D its density, g the force of gravity; then

$$S = \text{weight of one unit} = gD$$

$$M = \text{mass of } V \text{ units} = DV$$

$$W = \text{weight of } V \text{ units} = SV = gDV.$$

11. When a fluid acted on by gravity is at rest, the pressures are equal at all points in the same horizontal plane.

Let P, Q (fig. 4.) be any two points in the same horizontal plane in the interior of a mass of fluid at rest. The equilibrium of the fluid will not be disturbed if we enclose a part of it in a tube PMQ of uniform bore, having its branches MP, MQ symmetrical with respect to a vertical line. Then since the columns MP, MQ are symmetrical, and similarly situated with respect to the direction of gravity, they will balance when the pressures at P and Q are equal. But if the pressures at P and Q be unequal, the fluid will begin to move towards that end at which the pressure is the least, and the equilibrium will be destroyed; therefore in order that the fluid may be at rest, the pressures at P and Q must be equal.

12. To find the pressure at any point in a mass of fluid *at rest*.

Let the vertical prism AEF (fig. 5.) be a portion of a fluid at rest; then (Art. 4.) the equilibrium of AEF will not be disturbed if we suppose it to become solid. Now AEF is kept at rest by its own weight, and the pressures on its sides and ends. The pressures upon its sides act in a horizontal plane; its weight, and the pressures upon its ends, act in a vertical line; therefore the latter must be capable of maintaining equilibrium separately; therefore pressure on $DEF =$ weight of prism of fluid $AEF +$ pressure on ABC. Let m, p be the pressures at A, D respectively, ρ the density of the fluid, $ABC = \kappa$, $AD = z$; then the pressure on $ABC = m\kappa$, the pressure on $DEF = p\kappa$, and the weight of $AEF = g\rho z\kappa$; therefore $p\kappa = g\rho z\kappa + m\kappa$, therefore $p = g\rho z + m$.

Cor. 1. Let A be a point in the open surface of the fluid, then $m = 0$, and $p = g\rho z$.

Cor. 2. Since fluids press equally in all directions, and the pressure is the same at all points in the same horizontal plane, the pressure on a small area of any plane is ultimately equal to the pressure on an equal area of the horizontal plane that intersects it.

13. The surface of a fluid at rest is a horizontal plane.

Let A, P (fig. 6.) be any two points in the surface of a fluid at rest, AB, PQ vertical straight lines intersected by a horizontal plane in B, Q; ρ the density of the fluid. Then, (Arts. 11, 12.) $g\rho \,.\, PQ =$ pressure at $Q =$ pressure at $B = g\rho \,.\, AB$; therefore $PQ = AB$, therefore A and P are in the same horizontal plane.

14. The common surface of two fluids that do not mix is a horizontal plane.

Let A, P (fig. 7.) be any two points in the common surface of two fluids that do not mix; BAC, QPR vertical straight lines intersected by horizontal planes in B, Q, and in C, R; ρ, σ the densities of the upper and under fluids respectively. Then, (Art. 12.) pressure at $A -$ pressure at $B = g\rho \,.\, AB$, $CA =$

also pressure at $C -$ pressure at $A = g\sigma \,.\, AC$, $AB =$

$\therefore$ pressure at $C -$ pressure at $B = g \,.\, (\rho \,.\, AB + \sigma \,.\, AC)$

in like manner

press at C = press at R $\therefore$ $g\rho z + g\sigma y = g\rho z' + g\sigma y'$

$\therefore$ $z + y = z' + y'$.

$$\text{pressure at } R - \text{pressure at } Q = g\ (\rho . PQ + \sigma . PR),$$

and, (Art. 11.) pres. at Q = pres. at B, pres. at R = pres. at C,

$$\therefore\ \rho . PQ + \sigma . PR = \rho . AB + \sigma . AC,$$

$$\text{and } \sigma . PQ + \sigma . PR = \sigma . AB + \sigma . AC$$

$$\therefore\ (\sigma - \rho) . PQ = (\sigma - \rho) . AB,$$

$\therefore$ $PQ = AB$ $\therefore$ A and P are in the same horizontal plane.

Cor. Hence the surface of stagnant water exposed to the atmosphere is a horizontal plane.

15. If two fluids that do not mix, meet in a bent tube, the altitudes of their surfaces above the horizontal plane in which they meet, are inversely as their densities.

Let PAQ (fig. 8.) be a bent tube containing two fluids of different densities; AP, AQ the portions of the tube occupied by the lighter and heavier fluids; ρ, σ the densities of the fluids in AP, AQ. Let the planes of the surfaces of the fluids, and the plane in which they meet cut a vertical in H, K, C.

The pressure of the fluid in AP at $A = g\rho . HC$,

and the pressure of the fluid in AQ at $A = g\sigma . KC$.

When the fluids are in equilibrium these pressures must be equal

$$\therefore\ \rho . HC = \sigma . KC\ \therefore\ \frac{\rho}{\sigma} = \frac{KC}{HC}.$$

Cor. Let Π be the pressure of the atmosphere at the surface of each fluid. Then

pressure of fluid in AP at $A = \Pi + g\rho . HC$,

pressure of fluid in AQ at $A = \Pi + g\sigma . KC$,

and as before these pressures must be equal;

$$\therefore\ \rho . HC = \sigma . KC\ \therefore\ \frac{\rho}{\sigma} = \frac{KC}{HC}.$$

16. To find the pressure of a fluid on any surface.

Let BPC (fig. 9.) be the given surface. Draw AK vertical cutting the surface of the fluid in A, through H, K draw horizontal planes cutting the surface BPC in the curves PM, QN.

Let P be the pressure on MPB, S the area of MPB, ρ the density of the fluid, X the depth of the centre of gravity of BPC below the surface of the fluid, $AH=x$, $HK=\delta x$, therefore ultimately pressure on $MQ=d_x P.\delta x$, area $MQ=d_x S.\delta x$. But, ultimately,

$$\text{pressure on } MQ=g\rho.AH.MQ=g\rho.x.d_x S.\delta x,$$

$$\therefore d_x P=g\rho.x.d_x S \quad \therefore P=g\rho.\textstyle\int_x x.d_x S,$$

and the pressure on the whole surface $BPC=g\rho.\int_x x.d_x S$, the integral being taken between the limits corresponding to the highest and lowest points in the surface.

But $X.(\text{area } BPC)=\int_x x.d_x S$ between the same limits;

$$\therefore \text{pressure on } BPC=g\rho X.(\text{area } BPC);$$

or, the pressure of a fluid on any surface is equal to the weight of a column of the fluid whose base is equal to the area of the surface, and altitude equal to the depth of the centre of gravity of the surface below the surface of the fluid.

Cor. When the surface BPC is a plane, the pressures are all perpendicular to BPC, and consequently parallel to each other; therefore the resultant of the pressure on PBC is equal to the whole pressure, and acts in a direction perpendicular to BPC.

17. The centre of pressure of a plane surface immersed in a fluid is the point in which the resultant of the pressure of the fluid meets the surface.

To find the centre of pressure of any plane surface.

Let ABC (fig. 10.) be the surface, OY the line in which its plane cuts the surface of the fluid. From O draw OX in the plane ABC perpendicular to OY, and let X, Y be the co-ordinates of the centre of pressure referred to the axes OX, OY.

Then since the pressures are parallel to each other, we shall have, (Whewell's Mechanics Art. 84.)

$X.(\text{pressure on } ABC)=\text{moment of pressure on } ABC \text{ round } OY,$

$Y.(\text{pressure on } ABC)=\text{moment of pressure on } ABC \text{ round } OX,$

Draw MP, NQ parallel to OX; HP, KQ parallel to OY; PT perpendicular to the surface of the fluid meeting it in T.

Let $OH = x$, $HK, = \delta x$, $OM = y$, $MN = \delta y$, $TMP = \theta$, the density of the fluid $= \rho$. Therefore we have ultimately pressure on $PQ = g\rho . PT . PQ = g\rho . \sin\theta . x . \delta x . \delta y$;
moment of the pressure on PQ round OY

$$= g\rho . MP . PT . PQ = g\rho . \sin\theta . x^2 . \delta x . \delta y;$$

moment of pressure on PQ round OX

$$= g\rho . HP . PT . PQ = g\rho . \sin\theta . xy . \delta x . \delta y;$$

$$\therefore \text{ pressure on } ABC = g\rho . \sin\theta . \textstyle\int_x \int_y x;$$

moment of the pressure on ABC round $OY = g\rho . \sin\theta . \int_x \int_y x^2$;
moment of the pressure on ABC round $OX = g\rho . \sin\theta . \int_x \int_y xy$;
the integrals being taken between the limits corresponding to the boundary of the surface.

$$\therefore X . \textstyle\int_x \int_y x = \int_x \int_y x^2, \quad Y . \int_x \int_y x = \int_x \int_y xy.$$

Cor. 1. A physical plane ABC, one side of which is exposed to the pressure of a fluid, may be kept at rest by a single force equal and opposite to the pressure of the fluid applied at its centre of pressure.

Cor. 2. If ABC were a plane lamina of very small uniform thickness, moveable round the axis OY, the values of X and Y would be those of the co-ordinates of its centre of percussion.

18. To find the vertical pressure of a fluid on any surface.

Let ABR (fig. 11.) be a vertical cylindrical column of fluid in a mass of fluid at rest, meeting the surface of the fluid in ABC, and the given surface in PQR. The equilibrium of ABR will not be disturbed if we suppose it to become solid. Then since the vertical pressure on PQR, and the weight of ABR are the only forces that act vertically on ABR, the vertical pressure on PQR is equal to the weight of ABR, and acts in a vertical through the centre of gravity of ABR.

The vertical pressure upwards on PQR when it forms the under surface of the solid APQ, is equal to the vertical pressure downwards on PQR when it forms the upper surface of the solid PQD.

For the pressures on any portion of PQR are the same in either case, and they act in the same line but in opposite directions; therefore the vertical pressures are equal and act in opposite directions. Consequently the whole vertical pressures on PQR are equal and act in opposite directions.

Hence the vertical pressure of a fluid on any portion of the interior of the vessel in which it is contained, is equal to the weight of the superincumbent column of fluid.

19. To find that part of the pressure of a fluid on any surface which acts in a direction perpendicular to a given vertical plane.

Let ABR (fig. 12.) be a cylindrical column of fluid in a mass of fluid at rest, perpendicular to the given plane. Let it meet the given surface in PQR, and the vertical plane in ABC. Suppose ABR to become solid. Then since the resolved part of the pressure on PQR perpendicular to ABC, and the pressure on ABC are the only forces that act on ABR in a direction perpendicular to ABC, the resolved part of the pressure on PQR perpendicular to ABC is equal to the pressure on ABC, and acts in a line passing through the centre of pressure of ABC.

20. When the fluid sustains a pressure arising from the weight or elasticity of a lighter fluid resting upon its surface, we must suppose the pressure of the lighter fluid removed, and the depth of the heavier increased, so that the pressure at any given point beneath its original surface may remain unaltered. The amount of the pressure of the fluid on any surface estimated in a given direction, and the line in which its resultant acts may then be determined as in the preceding Articles.

21. To find the resultant of the pressure of a fluid on the surface of a solid immersed in it.

Since any portion V of a fluid at rest may become solid without disturbing the equilibrium of the fluid, the resultant of the pressure of the fluid on the surface of V after it has become solid, must be equal and opposite to the weight of V. But the fluid will exert the same pressure on the surface of any other solid of the same form as V, and occupying its place. And

the weight of V acts downwards in a vertical through its centre of gravity. Hence the resultant of the pressure of a fluid on the surface of a solid immersed in it is equal to the weight of the fluid displaced, and acts upwards in a vertical through the centre of gravity of the fluid displaced.

The solid may be either wholly or partly immersed, and the fluid of uniform or variable density.

Hence the resultant of the pressure of a fluid on the interior of the vessel in which it is contained is equal to the weight of the fluid and acts downwards in a vertical through its centre of gravity.

22. To find the conditions of equilibrium of a solid suspended in a fluid by a string.

Let GN, FM (fig. 13.) be verticals through the centres of gravity of the solid, and of the fluid displaced by it, EL the direction of the string by which the solid is suspended, T the tension of the string, W the weight of the solid, V the volume of the fluid displaced, ρ its density, and therefore $g\rho V$ the weight of the fluid displaced, or (Art. 21.) the resultant of the pressure of the fluid on the solid. Now W acts downwards in NG, $g\rho V$ acts upwards in FM; hence in order that the solid may be kept at rest by T acting in EL, EL must be vertical, and in the same plane with FM, GN; $T = W - g\rho V$ acting upwards, or $g\rho V - W$ acting downwards, according as W is greater or less than $g\rho V$; and if EGF be drawn perpendicular to GN in the plane GFN, $W . GE = g\rho V . FE$.

Cor. 1. W acting downwards in NG, and $g\rho V$ acting upwards in FM may be resolved into a single force $W - g\rho V$ acting downwards in NG, and a "couple" $g\rho V . FG$ in the plane MGF tending to make the solid revolve in the direction GFM; hence if any forces acting on the solid, can be resolved into a single force $W - g\rho V$ acting upwards in GN, and a "couple" $g\rho V . FG$ in the plane MGF tending to make the solid revolve in the direction MFG, they will keep it at rest.

Cor. 2. When a solid floats in equilibrium, it is kept at rest by its own weight acting downwards in a vertical through

its centre of gravity, and the weight of the fluid displaced acting upwards in a vertical through the centre of gravity of the fluid displaced; hence the weight of the fluid displaced must be equal to the weight of the solid, and the line joining the centres of gravity of the solid and of the fluid displaced must be vertical.

23. To find the positions in which a solid can float in equilibrium.

Let $f(x, y, z) = 0$ be the equation to the surface of the solid, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ the equation to the surface of the fluid, the centre of gravity of the solid being the origin of the co-ordinates: V the volume of the fluid displaced by the solid; X, Y, Z the co-ordinates of the centre of gravity of the fluid displaced; ρ the density of the fluid; W the weight of the solid. Then $g\rho V$ will be the weight of the fluid displaced, and $\frac{x}{X} = \frac{y}{Y} = \frac{z}{Z}$ the equations to the line joining the centres of gravity of the solid and of the fluid displaced. But when the solid is at rest, its weight is equal to the weight of the fluid displaced, and the line joining the centres of gravity of the solid and of the fluid displaced is perpendicular to the surface of the fluid, therefore

$$W = g\rho V, \text{ and } aX = bY = cZ. \quad \text{Also}$$

$$V = \int_x\int_y\int_z 1, \quad V . X = \int_x\int_y\int_z x, \quad V . Y = \int_x\int_y\int_z y, \quad V . Z = \int_x\int_y\int_z z.$$

The limits of the integrations being determined by the equations

$$f(x, y, z) = 0, \text{ and } \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

And having found the different values of a, b, c, we know the equation to the surface of the fluid corresponding to each position of equilibrium of the solid.

The section of a solid floating in equilibrium made by the surface of the fluid, is called the plane of floatation.

24. A solid floating in equilibrium is slightly elevated or depressed, and then left to itself; to determine its motion.

Let ADB (fig. 14.) be the position of the solid at the end of the time t from the beginning of its motion, CP a vertical meeting the surface of the fluid aCb in C and the plane of floatation APB in P, $PC = x$, a the space through which the solid was elevated or depressed, A the area of the plane of floatation, V the volume of the fluid displaced by the solid when at rest, ρ the density of the fluid: then the moving force on the solid in the direction PC will be the difference between its weight and the weight of the fluid displaced $= g\rho A.CP$, and the mass of the solid $= \rho V$, therefore the accelerating force on the solid in the direction $PC = g\dfrac{A}{V}.CP$

$$\therefore\ d_t^2 x + g\frac{A}{V}x = 0$$

$$\therefore\ x = a.\cos\sqrt{\left(g\frac{A}{V}\right)}.t.$$

Hence it appears that the body will oscillate vertically, the time of an oscillation being $\pi\sqrt{\left(\dfrac{V}{g.A}\right)}$.

25. To determine whether the equilibrium of a solid is stable or unstable.

Let the equilibrium of the solid be slightly disturbed by making it revolve through a very small angle in a vertical plane, without altering the quantity of fluid displaced; then the equilibrium of the solid will be stable or unstable, according as the pressure of the fluid tends to make it return to, or recede farther from its original position, that is according as a force acting upwards in a vertical through the centre of gravity of the fluid displaced, tends to diminish or increase the angle through which the solid has revolved.

26. When the equilibrium of a solid is slightly disturbed as in the preceding Article; to find the vertical through the centre of gravity of the fluid displaced.

Let G, H (fig. 15.) be the centres of gravity of the solid and of the fluid displaced by it, when floating in equilibrium. Let a plane through GH meet the plane of floatation in ACB, and the surface of the solid in ADB. Suppose the solid to

revolve through a very small angle θ in the plane ADB, so that the quantity of fluid displaced may be the same as before; and let ADB meet the surface of the fluid in aCb. Draw MF vertical through the centre of gravity of the fluid displaced by the solid in its new position, and mp, nq vertical through the centres of gravity of the wedges ACa, BCb. Then if the plane of floatation be symmetrical with respect to the plane ADB, mp, nq and consequently MF will be in the plane ADB. Draw HFE parallel to ab. Then from the two ways of making up the solid ADb,

$$(\text{vol. } aDb).FE + (\text{wedge } ACa).Cm$$
$$= (\text{vol. } ADB).HE - (\text{wedge } BCb).Cn.$$

And if x, y be the co-ordinates of any point in the boundary of the plane of floatation, ACB being the axis of x, and CY (perpendicular to ACa) the axis of y,

$$(\text{wedge } ACa).Cm = 2\theta \textstyle\int_x x^2 y, \text{ from } C \text{ to } A,$$
$$(\text{wedge } BCb).Cn = 2\theta \textstyle\int_x x^2 y, \text{ from } C \text{ to } B.$$

But $2\int_x x^2 y$ from C to $A + 2\int_x x^2 y$ from C to $B = 2\int_x x^2 y$ from A to $B = k^2 A$, $k^2 A$ being the moment of inertia of the plane of floatation round CY,

$$\therefore\ (\text{wedge } ACa).Cm + (\text{wedge } BCb).Cn = \theta k^2 A.$$

And wedge $ACa =$ wedge BCb, therefore

$$2\theta \textstyle\int_x xy \text{ from } C \text{ to } A = 2\theta \textstyle\int_x xy \text{ from } C \text{ to } B,$$

therefore C is the centre of gravity of the plane of floatation.

Also if the volume of the fluid displaced $= V$,

$$(\text{vol. } ADB).HE - (\text{vol. } aDb).FE = V.HF,\ HF = HM.\theta,$$
$$\therefore\ V.HM = k^2 A.$$

The point M, in which FM ultimately cuts HG is called the metacentre.

A force acting in the direction FM will tend to diminish or increase the angle HMF according as M is above or below G, therefore the equilibrium of the solid is stable or unstable according as M is above or below G.

If the plane of floatation be not symmetrical with respect to ADB, let aYb (fig. 16.) be the section of the solid made by the surface of the fluid; and let H, G, &c. be the projections of H, G, &c. in (fig. 15.) on the plane aYb. Draw pr, qs, MN

perpendicular to ab. It may be proved as before, that the centre of gravity of the plane of floatation lies in CY, and that $V.HN = k^2 A\theta$,

$$\text{Also } V.MN + (\text{wedge } YaY').pr = (\text{wedge } YbY').qs;$$

$$(\text{wedge } YaY').pr = \theta.\textstyle\int_x\int_y xy, \text{ from } C \text{ to } a;$$

$$(\text{wedge } YbY').qs = \theta.\textstyle\int_x\int_y xy, \text{ from } C \text{ to } b;$$

$$\therefore (\text{wedge } YbY').qs - (\text{wedge } YaY').pr$$

$$= \theta.\textstyle\int_x\int_y xy \text{ from } C \text{ to } b - \theta.\int_x\int_y xy \text{ from } C \text{ to } a$$

$$= \theta.\textstyle\int_x\int_y xy, \text{ from } a \text{ to } b; \ \therefore \text{ if } T = {}_{x}\int_y xy \text{ from } a \text{ to } b$$

$$V.MN = T\theta.$$

And the equilibrium will be stable or unstable according as H and G lie on the same or opposite sides of MN.

27. To determine the small oscillations of the solid DC (fig. 15.) when left to itself, after its equilibrium has been slightly disturbed, the solid being symmetrical with respect to the plane ADB.

Let the figure represent the position of the solid at the end of the time t from the beginning of the motion; and let ρ be the density of the fluid, K the radius of gyration of the solid revolving round G in the plane ADB, $HG = c$; then, retaining the notation of Art. 26., the moment of the pressure of the fluid tending to turn the solid round G in the direction FMG

$$= g\rho V.GM.\theta = g\rho V.\left(k^2 \frac{A}{V} - c\right).\theta,$$

and the moment of inertia of the solid round G in the plane $ADB = K^2 \rho V$,

$$\therefore d_t^2\theta + \frac{g}{K^2}.\left(k^2\frac{A}{V} - c\right).\theta = 0,$$

$$\text{if } \theta = \alpha \text{ when } d_t\theta = 0$$

$$\theta = \alpha.\cos\frac{1}{K}\sqrt{\left\{g\left(k^2\frac{A}{V} - c\right)\right\}}.t$$

Hence the time of an oscillation $= \pi \dfrac{K}{\sqrt{\left\{g\left(k^2\frac{A}{V} - c\right)\right\}}}$.

SECTION III.

ON THE EQUILIBRIUM OF ELASTIC FLUIDS ACTED ON BY GRAVITY.

ART. 28. To measure the pressure of the atmosphere.

Let a glass tube ABC (fig. 17.) closed at the end A, be bent at B, so that the branches AB, BC may be parallel and AB about thirty one inches longer than BC. Then if AB and part of BC be filled with mercury, and placed in a vertical position, the mercury will rise in BC, and sink in AB, (leaving a vacuum in the upper part of the tube,) till the pressure of the mercury at the common surface of the air and mercury in BC is equal to the pressure of the atmosphere. Let P, Q be points in the upper and lower surfaces of the mercury; through P, Q draw horizontal planes cutting a vertical HK in H and K. Let Π be the pressure of the atmosphere, σ the density of the mercury; then (Art. 12. Cor. 1.) the pressure of the mercury at $Q = g\sigma . HK$; and this must be equal to the pressure of the atmosphere at Q when the mercury is at rest,

$$\therefore \ \Pi = g\sigma . HK.$$

An instrument of this description furnished with a scale for measuring HK, is called a barometer.

29. The expansion of mercury between the temperatures of melting snow and boiling water is $\frac{10}{555}$ of its volume at the former temperature, and the increment of its volume is very nearly proportional to the increment of its temperature. Hence if σ_0, σ_t be the densities of mercury at 0°, t° (Centigrade) respectively, $\sigma_t . \left(1 + \frac{t}{5550}\right) = \sigma_0$.

If HK (fig. 17.) $=h$, t the temperature of the mercury in ABC, $\Pi = g\sigma_t . h = g\sigma_0 h \div \left(1 + \frac{t}{5550}\right) = g\sigma_0 h . \{1 - (0{,}00018)t\}$ nearly.

At the level of the sea in latitude 50^0 the mean value of h is 30,035 inches, the temperature of the air being $12^0.2$.

30. The pressure of air at a given temperature varies inversely as the space it occupies.

(I.) Let ABD (fig. 18.) be a glass tube having an open capillary termination at A, and bent at B so that the branches AB, BD may be parallel, and PC the axis of AB vertical. Pour mercury into DB till it rises to P, cutting off the communication between AB and BD, and then seal the aperture at A. Let mercury be now poured in till the horizontal plane through the surface of the mercury in BD meets PC in any point C; and let the surface of the mercury in AB meet PC in M. Then if h be the altitude of the mercury in the barometer, and σ its density at the time of making the experiment; u, v the capacities of the portions AM, AP of the tube; M, Π the pressures of the air in the tube when occupying the spaces u, v: we have (Arts. 29. 12.)

$\Pi =$ pressure of the exterior air $= g\sigma h$, $\mathrm{M} = g\sigma h + g\sigma . CM$.
But if u, v be measured, it will be found that

$$v : u = h + MC : h, \quad \therefore \mathrm{M} : \Pi = v : u.$$

(II.) Let AB (fig. 19.) be a glass tube having an open capillary termination at A; P, C any two points in its axis. Immerse AB vertically in mercury till the surface of the mercury meets PC in P, and seal the aperture at A. Elevate the tube till the plane of the surface of the mercury outside meets PC in C; and let the surface of the mercury within the tube meet PC in M. Then as before if u, v be the capacities of the portions AM, AP of the tube; M, Π the pressures of the air in the tube when occupying the spaces u, v: we have

$\Pi =$ pressure of the exterior air $= g\sigma h$, $\mathrm{M} = g\sigma h - g\sigma . C.M$.
And if u, v be measured, it will be found that

$$v : u = h - C.M : h, \quad \therefore \mathrm{M} : \Pi = v : u.$$

Hence the pressure of air at a given temperature varies inversely as the space it occupies, when the pressure is less than that of the atmosphere, as well as when it is greater.

31. Let ρ_0 be the density of atmospheric air under the pressure Π, at the temperature of melting snow; then since the pressure of air at a given temperature varies inversely as the space it occupies, and therefore directly as its density, $\Pi = \mu\rho_0$, where μ is constant.

If h be the altitude of the mercury in the barometer, σ its density, $\Pi = g\sigma h$, therefore $g\sigma h = \mu\rho_0$.

It is found that $\frac{\sigma_0}{\rho_0} = 10467$, σ_0 being the density of mercury at 0^0, when $h = 0,76$ metres $= 29,9218$ inches, and $g = 9,8088$ metres $= 386,18$ inches,

$$\therefore \sqrt{(\mu)} = 279,33 \text{ metres} = 916,46 \text{ feet}.$$

32. The expansion of air between the temperatures of melting snow and boiling water, under a constant pressure, is equal to 0,375 of its volume at the temperature of melting snow, and the increment of its volume is proportional to its temperature above 0^0, as indicated by a mercurial thermometer. Hence if ρ_0, ρ_{T} be the densities of air at the temperatures 0^0, T^0 under the same pressure, $\rho_0 = \{1 + (0,00375)\text{T}\}\rho_{\text{T}}$, therefore

$$\Pi = \mu . \{1 + (0,00375)\text{T}\}\rho_{\text{T}}.$$

33. When a given mass of atmospheric air is suddenly compressed or dilated, its temperature is increased or diminished according to the following law.

Let ρ, T^0; ρ_1, $\text{T}_1{}^0$ be the corresponding densities and temperatures of a given mass of air: then,

$$\text{T}_1 - \text{T} = (111,25) . \left(1 - \frac{\rho}{\rho_1}\right).$$

If, in making the first experiment described in Art. 30, the mercury be suddenly poured into the tube DB, the temperature

of the air in AB will be increased; and if the altitude CM (fig. 18.) be observed before the air in AB has cooled down to its original temperature, the pressure will appear to vary in a higher inverse ratio than that of the first power of the space occupied by the air. The same observation applies to the second experiment.

34. It appears from reasoning similar to that employed in Art. 11. that when an elastic fluid of uniform temperature, acted on by gravity, is at rest, its pressure, and therefore its density is the same at all points in the same horizontal plane. This is also true when the temperature at any point depends only on the distance of the point from a given horizontal plane.

35. To find the difference of the altitudes of two stations by means of the barometer.

Let M, Π be the pressures of the air at the points M, P in the vertical QPM (fig. 20.); S the temperature at M, T the temperature at P; Q a point very near to P; $MP = x$, $PQ = \delta x$; then $\Pi + d_x \Pi \, \delta x$ will ultimately be the pressure at Q. Now it is found that $\mathrm{S} - \mathrm{T}$ is proportional to x, we may therefore assume $\mathrm{S} - \mathrm{T} = cx$ or $\mathrm{T} = \mathrm{S} - cx$; therefore if g be the force of gravity, μ the ratio of the pressure of air to its density at 0°, E the expansion of air for one degree of heat under a constant pressure; we shall have the density of the air at P

$$= \frac{\Pi}{\mu} \cdot \frac{1}{1 + \mathrm{ET}} = \frac{\Pi}{\mu} \cdot \frac{1}{1 + \mathrm{ES} - \mathrm{E}cx}:$$

and the density between P and Q may be considered uniform, therefore ultimately (Art. 12.)

pressure at $P = g$(density at P)$.PQ$ + pressure at Q, or

$$\Pi = g \frac{\Pi}{\mu} \cdot \frac{\delta x}{1 + \mathrm{ES} - \mathrm{E}cx} + \Pi + d_x \Pi . \delta x,$$

$$\therefore \frac{1}{\Pi} \cdot d_x \Pi = - \frac{g}{\mu} \cdot \frac{1}{1 + \mathrm{ES} - \mathrm{E}cx} = - \frac{g}{\mu} \, \frac{1}{1 + \mathrm{ES}} \cdot (1 + \mathrm{E}cx)$$

nearly,

$$\therefore \log_\epsilon \Pi = C - \frac{g}{\mu} \cdot \frac{1}{1+\text{E S}} (x + \tfrac{1}{2}\text{E} c x^2)$$

$\log_\epsilon \text{M} = C$, since $\Pi = \text{M}$ when $x = 0$,

$$\therefore \log_\epsilon \frac{\text{M}}{\Pi} = \frac{g}{\mu} \cdot \frac{x}{1+\text{ES}} \cdot (1 + \tfrac{1}{2}\text{E} c x) = \frac{g}{\mu} \cdot \frac{x}{1+\text{E}(\text{S} - \frac{1}{2} c x)}$$

$$= \frac{g}{\mu} \cdot \frac{x}{1 + \frac{1}{2}\text{E}(\text{S}+\text{T})}.$$

Let h be the altitude, s the temperature of the mercury in the barometer at M; k the altitude, t the temperature of the mercury at any point in a horizontal plane passing through P; e the expansion of mercury for one degree of heat: then

$$\frac{\text{M}}{\Pi} = \frac{h}{k} \frac{1+et}{1+es}, \ \log_\epsilon \frac{\text{M}}{\Pi} = \log_\epsilon 10 . \log_{10} \frac{\text{M}}{\Pi}$$

$$= \log_\epsilon 10 . \{\log_{10} h - \log_{10} k - \log_{10}\epsilon . e(s-t)\},$$

$$\therefore x = \log_\epsilon 10 . \frac{\mu}{g} \{1 + \text{E}(\text{S}+\text{T})\} . \{\log_{10} h - \log_{10} k - \log_{10}\epsilon . e(s-t)\}.$$

$\log_\epsilon 10 . \frac{\mu}{g} = 60345 + 155 . \cos 2\lambda$ feet, λ being the latitude of the place of observation; $\frac{1}{2}\text{E} = 0.002$, $\log_{10}\epsilon . e = 0.00008$, the temperatures being expressed in degrees of the centigrade thermometer; $\{60345 + 155 . \cos 2\lambda\}\{1 + (0.002)(\text{S}+\text{T})\} = 60345 + 121(\text{S}+\text{T}) + 155 . \cos 2\lambda$ nearly; therefore if x be the number of feet contained in MP,

$$x = \{60345 + 121(\text{S}+\text{T}) + 155 \cos 2\lambda\}\{\log_{10} h - \log_{10} k - (0.00008)(s-t)\}$$

When MP is very small,

$$x = \{26207 + 52(\text{S}+\text{T})\}\left\{\frac{h-k}{h} - (0.00018)(s-t)\right\}.$$

36. When the difference of the altitudes of the stations is large, it becomes necessary to take into account the variation of gravity in the same vertical.

Let r be the distance of M (fig. 20.) from the centre of the earth, g the force of gravity at P, g' the force of gravity at M; then, retaining the notation of the preceding Article,

$$g = g'\left(\frac{r}{r+x}\right)^2, \text{ and}$$

$$\frac{1}{\Pi} d_x \Pi = -\frac{g}{\mu} \frac{1}{1 + \text{E}s - \text{E}cx} = -\frac{g'}{\mu}\left(\frac{r}{r+x}\right)^2 \frac{1}{1 + \text{E}s - \text{E}cx}$$

$$= -\frac{g'}{\mu} \frac{1}{1 + \text{E}s}\left(1 - 2\frac{x}{r} + \text{E}cx\right) \text{ nearly;}$$

$$\therefore \log_\epsilon \Pi = C - \frac{g'}{\mu} \frac{1}{1 + \text{E}s}\left(x - \frac{x^2}{r} + \tfrac{1}{2}\text{E}cx^2\right),$$

$$\log_\epsilon M = C; \; \therefore \; \log_\epsilon \frac{\text{M}}{\Pi} = \frac{g'}{\mu} \frac{1}{1 + \text{E}s}\left(x - \frac{x^2}{r} + \tfrac{1}{2}\text{E}cx^2\right)$$

$$= \frac{g'}{\mu} \frac{x}{1 + \frac{x}{r}} \frac{1}{1 + \text{E}(s - \frac{1}{2}cx)} = \frac{g'}{\mu} \frac{x}{1 + \frac{x}{r}} \frac{1}{1 + \frac{1}{2}\text{E}(s + \text{T})}$$

$$\text{And } \frac{\text{M}}{\Pi} = \frac{g'}{g} \frac{h}{k} \frac{1 + et}{1 + es} = \frac{h}{k}\left(1 + \frac{x}{r}\right)^2 \frac{1 + et}{1 + es};$$

$$\therefore \log_\epsilon \frac{\text{M}}{\Pi} = \log_\epsilon 10 . \log_{10} \frac{\text{M}}{\Pi}$$

$$= \log_\epsilon 10 \left(\log_{10} h - \log_{10} k - \log_{10} \epsilon . e(s - t) + \log_{10} \epsilon . 2\frac{x}{r}\right);$$

$$\therefore \; x = \log_\epsilon 10 . \frac{\mu}{g'}\left(1 + \frac{x}{r}\right)\left\{1 + \tfrac{1}{2}\text{E}(s + \text{T})\right\}$$

$$\left(\log_{10} h - \log_{10} k - \log_{10} \epsilon . e(s - t) + \log_{10} \epsilon . 2\frac{x}{r}\right);$$

$$\log_\epsilon 10 . \frac{\mu}{g'} = 60158 + 155 . \cos 2\lambda, \; 60158 \frac{1}{r} = 0{,}0029$$

$$\log_{10} \epsilon . e = 0{,}000078, \; \log_{10} \epsilon . \frac{2}{r} = 0{,}0000000416;$$

$$\therefore \; x = \left\{60158 + 120 . (s + \text{T}) + 155 . \cos 2\lambda + (0{,}0029)x\right\}$$
$$\left\{\log_{10} h - \log_{10} k - (0{,}000078) . (s - t) + (0{,}0000000416)x\right\}.$$

An approximate value of x must be first obtained from the equation

$$x=\{60345+121.(\mathrm{S}+\mathrm{T})\}\{\log_{10}h-\log_{10}k-(0{,}00008)(s-t)\},$$

and this substituted for x in the small terms will give a nearer value of x.

The values of $\frac{\mu}{g}$, and E are adapted to the mixture of air and watery vapour, constituting the atmosphere in its ordinary state. The vapour of water is lighter than air, under the same pressure, and the quantity of it contained in a given quantity of air increases with the temperature. Hence $\frac{\mu}{g}$, and E are larger than if the atmosphere consisted of perfectly dry air.

37. The pressure of vapour not in contact with the fluid from which it was produced, is found to be inversely proportional to the space it occupies, and its expansion, on being heated, is the same as that of air. If however the temperature, or the volume of a given quantity of vapour, be diminished beyond a certain point, a portion of it will return to the state of a liquid; and then, if the temperature of the vapour be invariable, its volume may be diminished till the whole becomes liquid, without increasing its pressure.

It appears probable from the experiments of Mr. Faraday, that every gas may be made to assume the form of a liquid by diminishing its volume. When the condensation of a gas is carried on nearly to the point at which it begins to liquefy, the ratio of its pressure to its density, at a given temperature is no longer constant. The value of this ratio for dry atmospheric air does not perceptibly change under the pressure of a column of mercury nearly ninety feet high.

SECTION IV.

ON THE EQUILIBRIUM OF FLUIDS ACTED ON BY ANY FORCES.

Art. 38. To find the pressure at any point in a mass of fluid at rest acted on by any forces.

Let PQ (fig. 21) be the edge of a very small prism of fluid in the interior of a mass of fluid at rest, R the accelerating force at P, S the resolved part of R in the direction PQ. Let the prism become solid; then (Art. 4) it will remain at rest; and since S. (mass prism), and the pressures on its ends are the only forces that act upon it in a direction parallel to PQ, they must be in equilibrium,

$\therefore$ press. on the end $Q -$ press. on the end $P = S$. (mass prism).

Let x, y, z; $x + \delta x$, $y + \delta y$, $z + \delta z$ be the co-ordinates of P, Q referred to rectangular axes Ox, Oy, Oz. Construct a parallelopiped LMN, of which PQ is the diagonal, having its edges PL, PM, PN parallel to Ox, Oy, Oz respectively. Let X, Y, Z be the components of R resolved parallel to Ox, Oy, Oz; κ the area of the base of the prism; ρ the density of the fluid; p the pressure at P, and therefore $p + d_x p \,.\, \delta x + d_y p \,.\, \delta y + d_z p \,.\, \delta z$, ultimately the pressure at Q. Then if the sides of the base of the prism be very small compared with its length, pressure on the end $Q -$ pressure on the end P

$$= \kappa \,(d_x p \,.\, \delta x + d_y p \,.\, \delta y + d_z p \,.\, \delta z).$$

$$S = X \,.\, \cos QPL + Y \,.\, \cos QPM + Z \,.\, \cos QPN,$$

and the mass of the prism $= \rho\kappa \,.\, PQ$, $\therefore S$. (mass prism)

$$= \rho\kappa \,.\, PQ \,.\, (X \,.\, \cos QPL + Y \,.\, \cos QPM + Z \,.\, \cos QPN)$$

$$= \rho\kappa \,(X \,.\, \delta x + Y \,.\, \delta y + Z \,.\, \delta z),$$

$$\therefore\ d_x p \,.\, \delta x + d_y p \,.\, \delta y + d_z p \,.\, \delta z = \rho\,(X \,.\, \delta x + Y \,.\, \delta y + Z \,.\, \delta z),$$

and δx, δy, δz are independent of each other;

$$\therefore\ d_x p = \rho X,\ \ d_y p = \rho Y,\ \ d_z p = \rho Z;$$

if, then, we can find a quantity p, such that

$$d_x p = \rho X,\ \ d_y p = \rho Y,\ \ d_z p = \rho Z,$$

p, taken between the proper limits, is the pressure at P.

39. When the fluid is elastic $p = \mu \rho$,

$$\therefore\ \frac{\mu}{\rho} \,.\, d_x \rho = X,\ \ \frac{\mu}{\rho}\, d_y \rho = Y,\ \ \frac{\mu}{\rho}\, d_z \rho = Z,$$

or ρ is a quantity such that

$$\mu d_x \log_\epsilon \rho = X,\ \ \mu d_y \log_\epsilon \rho = Y,\ \ \mu d_z \log_\epsilon \rho = Z,$$

and if u be a quantity such that $d_x u = X$, $d_y u = Y$, $d_z u = Z$,

$$\rho = C\epsilon^{\frac{u}{\mu}}.$$

40. $d_y d_x p = d_y (\rho X)$, $d_x d_y p = d_x (\rho Y)$; and $d_y d_x p = d_x d_y p$;
$\therefore\ d_y (\rho X) = d_x (\rho Y)$,

similarly $d_z (\rho X) = d_x (\rho Z)$, and $d_z (\rho Y) = d_y (\rho Z)$.

If we perform the differentiations and eliminate ρ, we obtain

$$X(d_z Y - d_y Z) + Y(d_x Z - d_z X) + Z(d_y X - d_x Y) = 0,$$

this equation expresses the relation that must exist between the forces X, Y, Z, in order that the equilibrium may be possible.

When the density is constant,

$$d_y X = d_x Y,\ \ d_z X = d_x Z,\ \ d_z Y = d_y Z.$$

41. If c be the pressure at any point P in the fluid, $p = c$, in which z is an implicit function of x and y, is the equation to the surface of equal pressure passing through P,

The derived equations of $p = c$ are

$$d_x p + d_z p \,.\, d_x z = 0,\ \ d_y p + d_z p \,.\, d_y z = 0,$$

x, y, z being considered independent of each other in forming the differential coefficients $d_x p$, $d_y p$, $d_z p$;

$$\therefore\ X + Z \,.\, d_x z = 0,\ \ Y + Z d_y z = 0;$$

therefore if a, β, γ be the angles between Ox, Oy, Oz, and the normal to the surface of equal pressure passing through P, at P,

$$\cos a = \frac{X}{R}, \quad \cos\beta = \frac{Y}{R}, \quad \cos\gamma = \frac{Z}{R}.$$

But $\frac{X}{R}$, $\frac{Y}{R}$, $\frac{Z}{R}$ are the cosines of the angles between Ox, Oy, Oz, and the direction in which the force at P acts. Therefore the force at any point acts in the direction of a normal to the surface of equal pressure at that point.

The equation to the surface of a fluid, is $p = 0$. And if the fluid be contiguous to another fluid with which it does not mix, and which exerts a pressure Π at the common surface of the fluids, $p = \Pi$ will be the equation to the common surface of the fluids.

42. When ρ is variable, and a quantity u can be found, such that $X = d_x u$, $Y = d_y u$, $Z = d_z u$, ρ must be a function of u. For

$$d_x p = \rho d_x u, \quad d_y p = \rho d_y u, \quad d_z p = \rho d_z u,$$

and these equations cannot be satisfied unless ρ is a function of u.

Let $\rho = fu$, then $d_x p = fu d_x u$, $\therefore p = \int_u fu$. Hence u and ρ are functions of p; and when p is constant, u and ρ must be constant; therefore $u = c$ is the equation to a surface of equal pressure; also ρ is the same at all points in a surface of equal pressure.

Hence when an elastic fluid of variable temperature is at rest, the temperature is the same at all points in a surface of equal pressure.

43. The conditions $d_y X = d_x Y$, $d_z X = d_x Z$, $d_y Z = d_z Y$, are satisfied whenever the forces tend to fixed centres, and the intensity of each force at any point P, is a function of the distance of P from the centre to which the force tends.

For if a, b, c be the co-ordinates of the centre to which one of the forces tends, r its distance from P, ϕr the intensity of the force at P,

$$X=\Sigma\left(\phi r\frac{x-a}{r}\right),\quad Y=\Sigma\left(\phi r\frac{y-b}{r}\right),\quad Z=\Sigma\left(\phi r\frac{z-c}{r}\right),$$

$$\text{and } r^2=(x-a)^2+(y-b)^2+(x-c)^2,$$

$$\therefore d_yX=\Sigma\left\{(d_r\phi r-\phi r)\frac{x-a}{r}\,\frac{y-b}{r}\right\},$$

we should have obtained the same expression for d_xY, therefore $d_yX=d_xY$, in like manner $d_zX=d_xZ$, and $d_yZ=d_zY$.

Cor. $u=\Sigma(\int_r\phi r)$. For if $u=\Sigma(\int_r\phi r)$,

$$d_xu=\Sigma\left(\phi r.\frac{x-a}{r}\right)=X,\quad d_yu=\Sigma\left(\phi r.\frac{y-b}{r}\right)=Y,$$

$$d_zu=\Sigma\left(\phi r\frac{z-c}{r}\right)=Z.$$

44. Each particle of a fluid attracts with a force which vanishes when the distance of the particle from the attracted point is finite; to find the pressure at any point in the interior of the fluid.

Let the plane xOy (fig. 22) be a tangent to the surface of the fluid touching it in O, Oz perpendicular to xOy, xOz, yOz the planes of greatest and least curvature through any point M in the surface of the fluid; draw NMQ parallel to Oz, meeting xOy in N.

Let R, S be the radii of curvature of the surface of the fluid in the planes xOz, yOz; $NOx=\theta$, $NO=\rho$, $NM=z$,

$$OP=r,\ MP=u,\ NQ=z,\ PQ=u_1;$$

ϕu the attraction of a portion of the fluid whose volume is unity on a point at the distance u, V the attraction of the fluid on P, which manifestly acts in the direction Oz.

The equation to the surface of the fluid is

$$z=\tfrac{1}{2}\rho^2\left\{\frac{(\cos\theta)^2}{R}+\frac{(\sin\theta)^2}{S}\right\},$$

$$\text{and } u_1^2=\rho^2+(z_1-r)^2,\quad u^2=\rho^2+(z-r)^2;$$

$$\therefore u_1\,.\,d_{z_1}u_1=z_1-r;$$

$$u^2=\rho^2+r^2-2rz \text{ nearly},$$

$$\therefore\ u d_\rho u = \rho - r d_\rho z = \rho \left\{ 1 - r \left(\frac{\overline{\cos\theta}|^2}{R} + \frac{\overline{\sin\theta}|^2}{S} \right) \right\};$$

$$\therefore\ \rho = u d_\rho u \left\{ 1 + r \left(\frac{\overline{\cos\theta}|^2}{R} + \frac{\overline{\sin\theta}|^2}{S} \right) \right\} \text{ nearly.}$$

$$V = (\textstyle\int_{\theta=2\pi} - \int_{\theta=0})(\int_{\rho=\infty} - \int_{\rho=0})(\int_{z_{,}=\infty} - \int_{z_{,}=z})\, \rho \dfrac{z_{,} - r}{u_{,}} \phi u_{,},$$

$$\textstyle\int_{z_{,}} \dfrac{z_{,} - r}{u_{,}} \phi u_{,} = \int_{z_{,}} \phi u_{,} d_{z_{,}} u_{,} = \int_{u_{,}} \phi u_{,};$$

when $z_{,} = \infty$, $u_{,} = \infty$; when $z_{,} = z$, $u_{,} = u$,

$$\therefore\ (\textstyle\int_{z_{,}=\infty} - \int_{z_{,}=z}) \dfrac{z_{,} - r}{u_{,}} \phi u_{,} = (\int_{u_{,}=\infty} - \int_{u_{,}=u}) \phi u_{,} = \varpi u.$$

$$\textstyle\int_\rho \rho \varpi u = \int_\rho u d_\rho u \left\{ 1 + r \left(\dfrac{\overline{\cos\theta}|^2}{R} + \dfrac{\overline{\sin\theta}|^2}{S} \right) \right\} \varpi u$$

$$= \left\{ 1 + r . \left(\frac{\overline{\cos\theta}|^2}{R} + \frac{\overline{\sin\theta}|^2}{S} \right) \right\} \textstyle\int_u u . \varpi u;$$

when $\rho = \infty$, $u = \infty$; when $\rho = 0$, $u = r$;

$$\therefore\ (\textstyle\int_{\rho=\infty} - \int_{\rho=0}) \rho \varpi u = \left\{ 1 + r \left(\dfrac{\overline{\cos\theta}|^2}{R} + \dfrac{\overline{\sin\theta}|^2}{S} \right) \right\} (\int_{u=\infty} - \int_{u=r}) u . \varpi u$$

$$= \left\{ 1 + r \left(\frac{\overline{\cos\theta}|^2}{R} + \frac{\overline{\sin\theta}|^2}{S} \right) \right\} \psi r.$$

$$(\textstyle\int_{\theta=2\pi} - \int_{\theta=0}) \left\{ 1 + r \left(\dfrac{\overline{\cos\theta}|^2}{R} + \dfrac{\overline{\sin\theta}|^2}{S} \right) \right\} = 2\pi \left\{ 1 + \dfrac{r}{2} \left(\dfrac{1}{R} + \dfrac{1}{S} \right) \right\};$$

$$\therefore\ V = 2\pi \left\{ 1 + \frac{r}{2} \left(\frac{1}{R} + \frac{1}{S} \right) \right\} \psi r.$$

Since the attraction of the fluid is insensible at sensible distances, ϖu decreases with extreme rapidity as u increases, and vanishes when the value of u becomes sensible. The same remark applies to ψr.

If the density of the fluid $= D$, the pressure at P arising from the attraction of the fluid $= D(\int_r - \int_{r=0}) V$,

$$= D\left\{2\pi(\int_r - \int_{r=0})\psi r + \pi\left(\frac{1}{R}+\frac{1}{S}\right)\right\}(\int_r - \int_{r=0}) r\psi r.$$

Let $2\pi(\int_{r=\infty} - \int_{r=0})\psi r = K$, $2\pi(\int_{r=\infty} - \int_{r=0}) r\psi r = H$, then since the force becomes insensible at sensible distances from the surface of the fluid, the pressure at P, arising from the attraction of the fluid, remains constant for all sensible values of OP,

$$\therefore\ 2\pi(\int_r - \int_{r=0})\psi r \text{ and } 2\pi(\int_r - \int_{r=0}) r\psi r$$

become K and H, as soon as r becomes sensible; therefore when OP is finite, the pressure at P, produced by the attraction of the fluid, $= D\left\{K + \frac{1}{2}H\left(\frac{1}{R}+\frac{1}{S}\right)\right\}$.

When the surface of the fluid is a plane, $\frac{1}{R} = 0$, $\frac{1}{S} = 0$, and the pressure $= DK$.

When the surface of the fluid is concave, R and S become negative.

Since ψr vanishes when r becomes sensible, $r\psi r$ is much less than ψr, therefore $\frac{1}{2}H\left(\frac{1}{R}+\frac{1}{S}\right)$ is much less than K, or, the attraction of a fluid on a particle of fluid in its surface, is nearly independent of the curvature of its surface.

45. Let ACD (fig. 23) be a narrow cylindrical tube, partly filled with a fluid acted on by no forces except its own attraction, and the attraction of the tube. Let mT be the attraction of the matter of which the tube is made on a particle of fluid in its surface, nT the attraction of the fluid on a particle of fluid in its surface, the surface in which the particle is placed being either a plane or a surface of continuous curvature.

(1) Let the surface of the fluid in the tube be a plane ABC perpendicular to the axis of the tube. Draw AD parallel to the axis of the tube, AC a diameter of the circle

ABC, and AG bisecting the angle CAD. The attraction of the tube on a particle of the fluid at A is equal to mT, and it acts in the direction CA. The attraction of the fluid $CBAD$ on a particle of the fluid at A is equal to $nT.\sin\frac{1}{4}\pi$; and the resolved parts of this attraction in the directions AD, AC are each equal to $nT(\sin\frac{1}{4}\pi)^2$, or $\frac{1}{2}nT$. But the whole attraction on a particle of the fluid at A, must be perpendicular to the surface of the fluid at A, or in the direction AD, therefore we must have $\frac{1}{2}nT = mT$, or $n = 2m$.

(2) Let the surface of the fluid in the tube be a concave hemisphere AEC. Complete the sphere $AECF$. The attraction on a particle of the fluid at A will not be sensibly altered if we suppose the upper part of the tube to be filled with fluid leaving the spherical space $AFCE$ vacant. But in order that the fluid surrounding $AFCE$ may be in equilibrium, the attraction on each particle in its surface must be the same, and perpendicular to the surface, therefore the attraction of the cylinder on a particle of the fluid at A must be equal to the attraction of the fluid on a particle of the fluid in its surface, or $nT = mT$, $\therefore\ n = m$.

When n is greater than m, it is probable that a layer of fluid adheres to the inner surface of the solid tube. The attraction of this fluid tube on a point in its surface is nT, and consequently the surface of the fluid contained in it is a concave hemisphere.

(3) Let the surface of the fluid in the tube be a convex hemisphere AFC. Complete the sphere $AFCE$. The attraction on a particle of the fluid at A will not be sensibly altered if we remove the fluid in $AECD$, leaving the sphere $AFCE$. But in order that the fluid sphere $AFCE$ may be in equilibrium, the attraction on a particle at A must be equal to the attraction on a particle at any point F in its surface. Therefore the attraction of the tube on a particle of the fluid at A must vanish, or $m = 0$.

The surface of water, alcohol, &c. contained in a glass tube of very small diameter is found to be a concave hemisphere.

The surface of mercury in such a tube is a convex hemisphere. The surface of mercury which has undergone a change in consequence of having been boiled for a long time in contact with atmospheric air, is a plane perpendicular to the axis of the tube. Tubes such as those mentioned above are called Capillary Tubes.

46. When the lower extremity of a capillary tube is immersed in fluid, the surface of the fluid within the tube is elevated above, or depressed below the surface of the surrounding fluid, according as it is concave or convex. Thus water is elevated, and mercury depressed in glass tubes. The attraction on which this phenomenon depends, is insensible at sensible distances: for the elevation or depression of the fluid is independent of the thickness of the tube; and the ascent of water in glass tubes is entirely prevented by a thin film of oil.

47. To determine the surface of a fluid contained in a vertical capillary tube.

Let AC (fig. 24.) be the axis of the tube meeting the plane of the surface of the exterior fluid in C; APB a section of the surface of the fluid, which will manifestly be a surface of revolution, and $BD\ B''D''$ a section of the tube made by a plane through AC; PN parallel to AC, AN perpendicular to AC; $DC = a$, $AC = c$, $AN = x$, $PN = y$; b the radius of curvature at A; R, S the radii of curvature at P in the plane APD, and perpendicular to APD; PEA a canal leading from P to A. Now when the fluid in PEA is at rest, the pressure at A produced by the action of gravity and the attraction at P on the fluid in PEA, must be equal to the pressure at A produced by the attraction at A on the fluid in AEP;

$$\therefore\ K - \tfrac{1}{2}H\left(\frac{1}{R} + \frac{1}{S}\right) + g.PN = K - \tfrac{1}{2}H\frac{2}{b};$$

$$\therefore\ 2\frac{g}{H}yx + \frac{2x}{b} = \frac{xd_x^2y}{\{1 + (d_xy)^2\}^{\frac{3}{2}}} + \frac{d_xy}{\{1 + (d_xy)^2\}^{\frac{1}{2}}};$$

$$\therefore\ 2\frac{g}{H}.\textstyle\int_x yx + \dfrac{x^2}{b} = \dfrac{xd_xy}{\{1 + (d_xy)^2\}^{\frac{1}{2}}} \qquad (1).$$

In order that the fluid in a canal leading from A to a point in the surface of the exterior fluid, may be in equilibrium, we must have $K - \frac{1}{2}H\frac{2}{b} + g.AC = K$, $\therefore\ H = gbc$;

$$\therefore\ \frac{g}{H}(cx^2 + 2\int_x yx) = \frac{x\,d_x y}{\{1+(d_x y)^2\}^{\frac{1}{2}}},$$

If α be the angle between a tangent to APB at B and AN, V the volume of the fluid in the tube elevated above the surface of the exterior fluid,

$$\frac{g}{H}\{ca^2 + 2(\int_{x=a} - \int_{x=o})yx\} = a\sin\alpha;$$

$$\text{and } V = \pi ca^2 + 2\pi(\int_{x=a} - \int_{x=o})yx;$$

$$\therefore\ V = \frac{H}{g}\pi a\sin\alpha.$$

α depends only on the nature of the fluid, and of the substance of which the tube is formed. When the fluid is water, and the tube of glass, $V = (0{,}023444)\,\pi a$, a being expressed in linear inches, and V in cubic inches. Also when a is small compared with c, the surface of the fluid is a concave hemisphere, $\therefore\ V = \pi a^2 c + \frac{1}{3}\pi a^3$, $\therefore\ ac + \frac{1}{3}a^2 = 0{,}023444$.

If $d_x y = \tan\theta$, and a be very small compared with c, $\int_x yx$ is small compared with cx^2, and

$$cx = \frac{H}{g}\sin\theta\,(1 + \frac{2}{cx^2}\int_x yx)^{-1} = \frac{H}{g}\sin\theta\,(1 - \frac{2}{cx^2}\int_x yx)$$

very nearly;

$$\text{And } cx = \frac{H}{g}\sin\theta,\ d_\theta y = \frac{H}{cg}\sin\theta,\ y = \frac{H}{cg}(1-\cos\theta) \text{ nearly.}$$

$$(\int_{x=a} - \int_{x=o})yx = (\int_{\theta=\alpha} - \int_{\theta=o})yx\,d_\theta x$$

$$= \left(\frac{H}{cg}\right)^3\{\tfrac{1}{2}(\sin\alpha)^2 - \tfrac{1}{3} + \tfrac{1}{3}(\cos\alpha)^3\};$$

$$\therefore\ ca = \frac{H}{g}\sin\alpha\,\{1 - \frac{2}{ca^2}\cdot\left(\frac{H}{cg}\right)^3(\tfrac{1}{2}(\sin\alpha)^2 - \tfrac{1}{3} + \tfrac{1}{3}(\cos\alpha)^3)\};$$

$$\therefore\ c = \frac{H}{g}\,\frac{\sin\alpha}{a}\left\{1 - \frac{a}{c\sin\alpha}\left(1 - \frac{2}{3}\,\frac{1-(\cos\alpha)^3}{(\sin\alpha)^2}\right)\right\} \quad (3) \text{ nearly.}$$

48. To determine the surface of a fluid between two parallel vertical plates.

Let $D'APD$ (fig. 24.) be a section of the surface of the fluid and of the parallel plates, made by a plane perpendicular to their surfaces; AC equidistant from the parallel plates, meeting the surface of the interior fluid in A, and the plane of the surface of the exterior fluid in C. Then, the rest of the construction and the notation being the same as in Art. 47.) we have

$$2\frac{g}{H}y + \frac{1}{b} = \frac{d_x^2 y}{\{1 + (d_x y)^2\}^{\frac{3}{2}}};$$

$$\therefore\ 2\frac{g}{H}\int_x y + \frac{x}{b} = \frac{d_x y}{\{1 + (d_x y)^2\}^{\frac{1}{2}}} \quad (4);$$

$$\text{and } H = 2gbc;$$

$$\therefore\ 2\frac{g}{H}(cx + \int_x y) = \frac{d_x y}{\{1 + (d_x y)^2\}^{\frac{1}{2}}};$$

$$\therefore\ 2\frac{g}{H}\{ca + (\int_{x=a} - \int_{x=0})y\}. = \sin\alpha$$

$$\text{or area } D'B'ADB = \frac{H}{g}\sin\alpha \qquad (5).$$

When the fluid is water, and the plates of glass, and very close to each other, $B''AB$ is a semicircle,

$$\therefore\ \text{area } D,\ B''ABD = 2ae + 2a^2 - \tfrac{1}{2}\pi a^2 = 2ac + \tfrac{3}{7}a^2 \text{ nearly,}$$

$$\text{and } \frac{H}{g} = 0.023444,\ \therefore\ 2ac + \tfrac{3}{7}a^2 = 0.023444.$$

If $d_x y = \tan\theta$, then, when a is small,

$$cx = \frac{H}{2g}\sin\theta\,(1 - \frac{1}{cx}\int_x y) \text{ very nearly; and}$$

$$(\int_{x=a} - \int_{x=0})y = (\int_{\theta=\alpha} - \int_{\theta=0})y\,d_\theta x$$

$$= \left(\frac{H}{2gc}\right)^2(\sin\alpha - \frac{\alpha}{2} - \tfrac{1}{4}\sin 2\alpha);$$

$$\therefore\ ca = \frac{H}{2g}\sin\alpha\left\{1 - \frac{1}{ca}\left(\frac{H}{2gc}\right)^2(\sin\alpha - \frac{\alpha}{2} - \tfrac{1}{4}\sin 2\alpha)\right\};$$

$$\therefore\ c = \frac{H}{g}\ \frac{\sin\alpha}{2\alpha}\left\{1 - \frac{\alpha}{c\,(\sin\alpha)^2}(\sin\alpha - \frac{\alpha}{2} - \tfrac{1}{4}\sin 2\alpha)\right\} \quad (6).$$

It appears that the elevation of a fluid between two parallel plates, is nearly half the elevation in a tube whose diameter is equal to the distance between the plates.

When a single plate is immersed vertically in a fluid, we have

$$2\frac{g}{H}y = \frac{d_x^{\,2}y}{\{1 + (d_x y)^2\}^{\frac{3}{2}}}.$$

This is the differential equation to the elastic curve.

The investigation of the form of the surface of the fluid when it is depressed, leads to precisely the same equations as when it is elevated, the sign of y being changed.

If V be the space between the surface of the mercury in a vertical glass tube and the plane of the surface of the mercury on the outside, and a the radius of the tube, $V = (0{,}01)\,\pi a$.

49. To find the tension of a flexible cylindrical vessel containing fluid.

Let MK, PQ, HL (fig. 25.) be equidistant sections of the cylinder made by planes perpendicular to its axis. Draw PE, QE normals at the extremities of the small arc PQ; MPH, KQL perpendicular to PEQ; and let p be the pressure of the fluid at P, $t.MH$ the tension of MH or KL, r the radius of curvature of PQ. Now ML is kept at rest by the pressure of the fluid, and the tensions of its edges; the tensions of MH, KL, and the pressure of the fluid, are the only forces that act in the plane PEQ; the tensions act perpendicular to PE, QE respectively, and the pressure of the fluid acts perpendicular to PQ, therefore ultimately

$$EP : PQ = t.MH : p.(\text{area } ML) = t : p.PQ; \quad \therefore\ t = pr.$$

50. To find the tension of a vessel of any form containing fluid.

Let PCP', QCQ' (fig. 26.) be the normal sections of least and greatest curvature of the vessel at C; $P'C = PC$, $Q'C = QC$; PE, $P'E$, QF, $Q'F$ normals at the extremities of the small arcs PCP', QCQ'; MPK, HPL sections of the vessel made by planes perpendicular to PEP'; MQH, KQL sections of the vessel made by planes perpendicular to QFQ'. Let p be the pressure of the fluid at C, $t.QQ'$ the tension of HL or MK, $v.PP'$ the tension of MH or KL; r, s the radii of curvature of PCP', QCQ'. ML is kept at rest by the pressure of the fluid, and the tensions of its edges; therefore the resultant of the tensions must be equal and opposite to the pressure of the fluid. The resultant of the tensions of MK, HL.

$$= 2t.QQ'.\sin PEC = \frac{t}{r}.PP'.QQ';$$

The resultant of the tensions of MH, KL

$$= 2v.PP'.\sin QFC = \frac{v}{s}.PP'.QQ'.$$

And the resultants of the tensions act in the direction CE; the pressure of the fluid $= p.PP'.QQ'$, and it acts in the direction EC,

$$\therefore\ p = \frac{t}{r} + \frac{v}{s}.$$

When the tensions are the same in every direction, or $v = t$,

$$p = t\left(\frac{1}{r} + \frac{1}{s}\right).$$

When the vessel is immersed in fluid, p is the difference of the pressures of the interior and exterior fluids.

SECTION V.

ON THE MOTION OF FLUIDS.

Art. 51. When an incompressible fluid flows through a tube, the velocities of the fluid at any two points, are inversely proportional to the areas of the perpendicular sections of the tube at those points; supposing the tube to continue always full, and the velocities at all points in the same section, to be equal to one another, and perpendicular to the section.

For equal volumes of fluid must pass through each section in the same time; and if u, v be the velocities at the two sections; H, K the areas of the sections; $u\mathrm{H}t$, $v\mathrm{K}t$ will be the volumes of the fluid that passes through the two sections in the small time t; and these are equal,

$$\therefore\ u\mathrm{H} = v\mathrm{K}, \text{ and } \therefore\ u : v = \mathrm{K} : \mathrm{H}.$$

52. When a fluid is in motion acted on by any forces, to determine the effective accelerating force in the direction of its motion at any point.

Draw the curve $APQR$ (fig. 27.) so that a tangent to it at any point may be in the direction of the motion of the fluid at that point. The motion of the fluid will not be altered if we suppose a portion of it, of the form of a very small cylinder having PQ for its axis, to become solid for an instant.

Let p be the pressure of the fluid at P, S the accelerating force at P resolved in the direction of a tangent to APR at P, $AP = s$; then $p + d_s p \,.\, PQ$ will ultimately be the pressure at Q; and if ρ be the density of the fluid at P, κ the area of the base of the cylinder PQ; the mass of $PQ = \rho\kappa \,.\, PQ$, and the moving force

on PQ in the direction $PQ = S.(\text{mass } PQ)$ + pressure on the end P − pressure on the end $Q = S\rho\kappa . PQ - \kappa d_s p . PQ$; therefore the effective accelerating force on the fluid at P in the direction PQ

$$= S - \frac{1}{\rho} d_s p.$$

53. To find the relation between the pressure and the velocity at P, when the velocity at any point is independent of the time.

Let v be the velocity at P, then $v d_s v$ = the effective accelerating force at P in the direction PQ,

$$\therefore v d_s v = S - \frac{1}{\rho} d_s p.$$

When the fluid is non-elastic $\frac{1}{2} v^2 + \frac{1}{\rho} p = \int_s S.$

When the fluid is elastic, $p = \mu \rho$,

$$\therefore v d_s v = S - \frac{\mu}{p} d_s p, \quad \therefore \frac{1}{2} v^2 + \mu \log_\epsilon p = \int_s S.$$

$\int_s S = \frac{1}{2} V^2 + C$, where V is the velocity acquired by a point in moving from A to P in a tube AP, acted on by the same forces as the fluid.

54. If the fluid be acted on by gravity only, and if z be the depth of P below a given horizontal plane,

$$S = g d_s z, \quad \int_s S = g z + C;$$

therefore when the fluid is non-elastic,

$$\frac{1}{2} v^2 + \frac{1}{\rho} p = g z + C:$$

and when the fluid is elastic $\frac{1}{2} v^2 + \mu \log_\epsilon p = g z + C.$

55. To find the relation between the pressure and the velocity at P, when the velocity depends upon the time as well as the position of P.

Let v be the velocity at P at the end of the time t, PR the space described by a particle of the fluid in the very small time δt, $v + \delta v$ the velocity at R. v is a function of s and t;

$$\therefore \ \delta v = d_t v . \delta t + d_s v . PR, \text{ ultimately.}$$

$$\text{But } \delta v = (S - \frac{1}{\rho} d_s p) \delta t \text{ ultimately;}$$

$$\therefore \ d_t v + d_s v \text{ limit } \frac{PR}{\delta t} = S - \frac{1}{\rho} d_s p, \text{ and limit } \frac{PR}{\delta t} = v;$$

$$\therefore \ d_t v + v d_s v = S - \frac{1}{\rho} d_s p.$$

When the fluid is acted on by gravity only,

$$d_t v + v d_s v = g d_s x - \frac{1}{\rho} d_s p.$$

56. To find the velocity with which an incompressible fluid acted on by gravity issues through an indefinitely small orifice in the vessel containing it. Let K (fig. 28.) be the orifice. Draw KH vertical meeting the plane of the surface of the fluid in H. Let $KH = h$; u the velocity of the fluid at K; p, v the pressure and velocity at any point P in the fluid, x the depth of P below the surface of the fluid; Π the pressure of the atmosphere. Then (Art. 54.)

$$\tfrac{1}{2} v^2 + \frac{1}{\rho} p = g x + C.$$

At the surface of the fluid $x = 0$, $p = \Pi$, and $v = 0$; (for (Art. 51.) velocity at the surface : u = area orifice : area surface, and the area of the orifice vanishes compared with the area of the surface of the fluid, therefore the velocity at the surface $= 0$.) $\therefore \ \frac{1}{\rho} \Pi = C$.

$$\text{At } K,\ x = h,\ p = \Pi,\ v = u,$$

$$\therefore \ \tfrac{1}{2} u^2 + \frac{1}{\rho} \Pi = g h + C;$$

$$\therefore \ \tfrac{1}{2} u^2 = g h.$$

Or, the velocity of the issuing fluid is equal to the velocity acquired by a heavy body in falling down HK.

COR. If M be the pressure of the fluid at any point in a horizontal plane meeting KH in H, Π the pressure of the atmosphere at K;

$$\frac{1}{\rho}\text{M} = C, \text{ and}$$

$$\tfrac{1}{2}u^2 + \frac{1}{\rho}\Pi = gh + C;$$

$$\therefore \tfrac{1}{2}u^2 = gh + \frac{1}{\rho}(\text{M} - \Pi).$$

When the fluid issues through an orifice in a thin plate, it does not acquire its greatest velocity till it reaches a point at a small distance from the orifice. This part of the stream is called the "vena contracta", on account of the contraction of the stream resulting from its increased velocity. The area of a section of the "vena contracta" is equal to about $\frac{5}{8}$ of the area of the orifice.

57. Let KT be the direction of the issuing fluid; HT the intersection of the plane HKT and the plane of the surface of the fluid; Kx parallel to HT; $TKx = a$. Then each drop of the issuing stream being projected in the direction KT with the velocity acquired by a heavy body in falling down HK, and being acted on by gravity, will describe a parabola having HT for its directrix.

The equation to the curve described by the issuing stream is

$$y = x.\tan a - \frac{g}{2u^2}x^2(\text{secant } a)^2: \text{ or,}$$

$$y = x.\tan a - \frac{x^2}{4h}(\text{secant } a)^2.$$

The velocity of the issuing fluid may be deduced from observed values of the angle TKx, and of the range of the stream on a horizontal plane at a given depth below the orifice. The value of the velocity determined in this manner is found to coincide very accurately with its theoretical value.

58. To find the time of emptying a vessel through a very small orifice.

Let κ be the area of the effective orifice, or of the section of the "vena contracta," x the depth of the orifice below the surface, X the area of the surface, u the velocity at the "vena contracta" at the end of the time t from the beginning of the motion. Then $-d_t x$ will be the velocity of the descending surface; and (Art. 51) $(-d_t x).\text{X} = u\kappa$; also if κ be very small compared with X, $u = \sqrt{(2gx)}$,

$$\therefore -d_t x.\text{X} = \kappa\sqrt{(2gx)}, \text{ and } d_t x.d_x t = 1;$$

$$\therefore \sqrt{(2g)}\kappa.d_x t + \frac{\text{X}}{\sqrt{(x)}} = 0.$$

It appears from experiment, that when the orifice is immersed in fluid, the quantity of fluid discharged in a given time, is the same as when the discharge takes place into air; the perpendicular distance between the surfaces of the fluids in the former case, being equal to the depth of the orifice below the surface in the latter.

59. An incompressible fluid acted on by gravity issues through a finite orifice in the horizontal base of the vessel in which it is contained; to determine its motion.

Draw AK (fig. 29.) vertical meeting the plane of the orifice in K. At the end of the time t from the beginning of the motion, let the surface of the fluid meet AK in H; and let p, v be the pressure and velocity at any point P in AK; $AK = c$; $AH = x$; $AP = z$; K the area of the orifice; X the area of the surface of the fluid; Z the area of a horizontal section of the vessel through P; Π the pressure of the atmosphere. Then since the fluid is acted on by gravity only (Art. 55.),

$$d_t v + v d_z v = g d_z z - \frac{1}{\rho} d_z p,$$

$$\therefore \int_z d_t v + \tfrac{1}{2} v^2 = gz - \frac{1}{\rho} p.$$

And if we suppose the motion of the fluid to be vertical, and the velocity, at a given instant, the same at all points in

the same horizontal section, $d_z s = 1$, and $Zv = \text{K}u$. Now Z is independent of t, and $d_t u$ is independent of z, $\therefore Z d_t v = \text{K} d_t u$,

$$\text{and } \int_s d_t v = \int_z d_t v \,.\, d_z s = \int_z d_t v = \int_z \frac{\text{K}}{Z} d_t u = \text{K} \,.\, d_t u \int_z \frac{1}{Z}.$$

Hence, the equation $\int_s d_t v + \frac{1}{2} v^2 = gz - \frac{1}{\rho} p$, becomes

$$\text{K} d_t u \int_z \frac{1}{Z} + \frac{1}{2} \frac{\text{K}^2}{Z^2} u^2 = gz - \frac{1}{\rho} p. \quad (1)$$

At H, $z = x$, $Z = \text{X}$, $p = \Pi$,

$$\therefore \text{K} \,.\, d_t u \int_{z=x} \frac{1}{Z} + \frac{1}{2} \frac{\text{K}^2}{\text{X}^2} u^2 = gx - \frac{1}{\rho} \Pi.$$

At K, $z = c$, $Z = \text{K}$, $p = \Pi$,

$$\therefore \text{K} \,.\, d_t u \int_{z=c} \frac{1}{Z} + \frac{1}{2} u^2 = gc - \frac{1}{\rho} \Pi.$$

$$\text{K} \,.\, d_t u \,.\, \left(\int_{z=c} - \int_{z=x}\right) \frac{1}{Z} + \frac{1}{2} \left(1 - \frac{\text{K}^2}{\text{X}^2}\right) u^2 = g\,(c - x). \quad (2)$$

The velocity of the surface of the fluid is $d_t x$, and $d_t u = d_x u \,.\, d_t x$, therefore when $d_t x$ is known in terms of x and u, (2) will give u and x in terms of t, and then p may be obtained from (1).

When the issuing stream is contracted, the section of the "vena contracta" must be considered as the orifice.

Cor. If the pressure at $H = \text{M}$, and the pressure at the orifice $= \Pi$,

$$\text{K} d_t u \left(\int_{z=c} - \int_{z=x}\right) \frac{1}{Z} + \frac{1}{2} \left(1 - \frac{\text{K}^2}{\text{X}^2}\right) u^2 = g\,(c - x) + \frac{1}{\rho} (\text{M} - \Pi).$$

60. When the vessel is continually supplied with fluid, so that the surface of the fluid may remain stationary,

$$x, \text{ and } \left(\int_{z=c} - \int_{z=x}\right) \frac{1}{Z} \text{ are constant.}$$

Hence if $K\lambda\,(\int_{z=c} - \int_{z=x})\,\frac{1}{Z} = \{2g\,(c-x)\}^{\frac{1}{2}}.\left(1-\frac{K^2}{X^2}\right)^{\frac{1}{2}}$,

$$\{2g(c-x)\}^{\frac{1}{2}} - u\left(1-\frac{K^2}{X^2}\right)^{\frac{1}{2}} = \left\{\{2g(c-x)\}^{\frac{1}{2}} + u\left(1-\frac{K^2}{X^2}\right)^{\frac{1}{2}}\right\}\epsilon^{-\lambda t}.$$

When K is small, and t finite, λt is very large, and therefore $\epsilon^{-\lambda t}$ is very small. Consequently, at the end of a finite time from the beginning of the motion,

$$u^2\left(1-\frac{K^2}{X^2}\right) = 2g\,(c-x) \text{ very nearly.}$$

When the velocity of the fluid at a given point, is independent of the time, $d_t u = 0$,

$$\therefore\ u^2\left(1-\frac{K^2}{X^2}\right) = 2g\,(c-x).$$

61. When the waste of the fluid is not supplied, $X.d_t x = Ku$, and equation (2) becomes

$$\frac{K^2}{X}.u d_x u\,(\int_{z=c} - \int_{z=x})\,\frac{1}{Z} + \tfrac{1}{2}\left(1-\frac{K^2}{X^2}\right)u^2 = g\,(c-x);$$

$$\therefore \text{ if } (\int_{z=c} - \int_{z=x})\,\frac{1}{Z} = N,$$

$$\tfrac{1}{2}K^2u^2 = \epsilon^{-\int_x\left(\frac{1}{K^2}-\frac{1}{X^2}\right)\frac{X}{N}}.\int_x g\,\frac{X}{N}(c-x)\,\epsilon^{\int_x\left(\frac{1}{K^2}-\frac{1}{X^2}\right)\frac{X}{N}}.$$

62. An incompressible fluid acted on by gravity flows through a tube; to determine the motion of the fluid.

Let APK (fig. 30.) be the axis of the tube; and, at the end of the time t from the beginning of the motion, let p be the pressure, and v the velocity at P; $AP = s$; z the depth of P below a horizontal plane through A; S the area of a section of the tube at P; u the velocity at any point K; K the area of a section of the tube at K.

Then since the fluid is acted on by gravity only, (Art. 55.)

$$d_t v + v\,d_s v = g\,d_s z - \frac{1}{\rho}\,d_s p;$$

$$\therefore\ \int_s d_t v + \tfrac{1}{2} v^2 = gz - \frac{1}{\rho} p.$$

$\mathrm{S}v = \mathrm{K}u$, and S is independent of t; therefore $\mathrm{S}.d_t v = \mathrm{K}.d_t u$ and $d_t u$ is independent of s,

$$\therefore\ \int_s d_t v = \mathrm{K}\, d_t u \int_s \frac{1}{\mathrm{S}};$$

$$\therefore\ \mathrm{K}\, d_t u \int_s \frac{1}{\mathrm{S}} + \tfrac{1}{2} \frac{\mathrm{K}^2}{\mathrm{S}^2} u^2 = gz - \frac{1}{\rho} p. \quad (1)$$

Let H, L be the extremities of the column of fluid; $AH = s_{,}$, $AL = s_{,,}$; $\mathrm{S}_{,}$, $\mathrm{S}_{,,}$ the areas of sections of the tube at H, L; $z_{,}$, $z_{,,}$ the depths of H, L below a horizontal plane through A. Then

$$\mathrm{K}\, d_t u \int_{s=s_{,}} \frac{1}{\mathrm{S}} + \tfrac{1}{2} . \frac{\mathrm{K}^2}{\mathrm{S}_{,}^2} u^2 = gz_{,} - \frac{1}{\rho} \Pi.$$

$$\mathrm{K}\, d_t u \int_{s=s_{,,}} \frac{1}{\mathrm{S}} + \tfrac{1}{2} \frac{\mathrm{K}^2}{\mathrm{S}_{,,}^2} u^2 = gz_{,,} - \frac{1}{\rho} \Pi.$$

$$\therefore\ \mathrm{K}.d_t u \left(\int_{s=s_{,,}} - \int_{s=s_{,}}\right) \frac{1}{\mathrm{S}} + \tfrac{1}{2} \left(\frac{1}{\mathrm{S}_{,,}^2} - \frac{1}{\mathrm{S}_{,}^2}\right) \mathrm{K}^2 u^2 = g\,(z_{,,} - z_{,}). \quad (2)$$

When the quantity of fluid in the tube is constant, let V be its volume, then

$$V = \left(\int_{s=s_{,,}} - \int_{s=s_{,}}\right) \mathrm{S}; \quad \mathrm{S}_{,}.d_t s_{,} = \mathrm{K}u; \quad \mathrm{S}_{,,}.d_t s_{,,} = \mathrm{K}u.$$

From these equations, equation (3), and the equations to the axis of the tube, we may obtain $s_{,}$, $s_{,,}$, u.

If K be the extremity of the tube, $AK = c$, a the depth of K below a horizontal plane through A;

$$\mathrm{K}\, d_t u \left(\int_{s=c} - \int_{s=s_{,}}\right) \frac{1}{\mathrm{S}} + \tfrac{1}{2} \left(1 - \frac{\mathrm{K}^2}{\mathrm{S}_{,}^2}\right) u^2 = g\,(a - z_{,}) \quad (3);$$

$\mathrm{S}_{,}.d_t s_{,} = \mathrm{K}u$; and when $d_t s_{,}$ is known in terms of $s_{,}$ and u, $s_{,}$, u may be found from (3) and then p may be found from (1).

63. To determine the velocity with which a small disturbance is propagated along a horizontal column of fluid.

Let AP (fig. 31.) be a horizontal tube filled with fluid, the equilibrium of which has been slightly disturbed; P, Q discs serving to separate the fluid between P and Q, from the rest of the fluid in the tube, without impeding its motion; κ the area of a section of the tube; x, and $x + \delta x$ the distances of P and Q from A, and ρ the density of the fluid, before its equilibrium was disturbed. At the end of the time t from the beginning of the motion, let $AP = y$, and the pressure at $P = p$; therefore $AQ = y + d_x y . \delta x$, and the pressure at $Q = p + d_x p . \delta x$ ultimately.

The moving force on the cylinder of fluid PQ in the direction AP = pressure on the end P − pressure on the end Q $= -\kappa d_x p . \delta x$; and the mass of the fluid in $PQ = \kappa\rho . \delta x$; therefore effective acc^g. force on PQ in the direction $AP = -\frac{1}{\rho} d_x p$,

$$\therefore\ d_t^2 y = -\frac{1}{\rho} d_x p.$$

(1) Let the fluid be air or gas; μ the ratio of its pressure to its density at 0^0; E its expansion for one degree of heat, $\mathrm{C}\left(1 - \frac{\rho}{\rho_{,}}\right)$ the number of degrees by which the temperature of a given mass of the fluid is increased, when its density is suddenly changed from ρ to $\rho_{,}$; T the temperature of the fluid when at rest; $\mathrm{T}_{,}$ the temperature of the fluid at P; $\rho_{,}$ the density of the fluid at P,

$$\therefore\ \mathrm{T}_{,} - \mathrm{T} = \mathrm{C}\left(1 - \frac{\rho}{\rho_{,}}\right).$$

The volume of $PQ = \kappa . d_x y . \delta x$, therefore the mass of the fluid in $PQ = \kappa\rho_{,} d_x y . \delta x$; but the mass of the fluid in PQ

$$= \kappa\rho . \delta x, \quad \therefore\ \rho_{,} . d_x y = \rho.$$

And $p = \mu(1 + \mathrm{E}\mathrm{T}_{,})\rho_{,} = \mu\left\{1 + \mathrm{E}(\mathrm{T}_{,} - \mathrm{T})\right\}(1 + \mathrm{E}\mathrm{T})\rho_{,}$

$$= \mu\left\{1 + \mathrm{E}\mathrm{C}\left(1 - \frac{\rho}{\rho_{,}}\right)\right\}(1 + \mathrm{E}\mathrm{T})\rho_{,}$$

$$= \mu(1 + \mathrm{E}\mathrm{C})(1 + \mathrm{E}\mathrm{T})\rho_{,} - (1 + \mathrm{E}\mathrm{T})\rho$$

$$= \mu(1 + \mathrm{E}\mathrm{C})(1 + \mathrm{E}\mathrm{T})\rho\frac{1}{d_x y} - (1 + \mathrm{E}\mathrm{T})\rho;$$

$$\therefore \frac{1}{\rho} d_x p = -\mu(1+\text{E c})(1+\text{E T}) \frac{d_x^2 y}{(d_x y)^2};$$

And since the disturbance is small, ρ and $\rho_{,}$ are very nearly equal, therefore $d_x y = 1$ very nearly;

$$\therefore \frac{1}{\rho} d_x p = -\mu(1+\text{E c})(1+\text{E T}) d_x^2 y;$$

$$\therefore d_t^2 y = \mu(1+\text{E c})(1+\text{E T}).d_x^2 y.$$

(2) Let the fluid be of the kind denominated non-elastic, or liquid. The increment of the density of a liquid under a moderate pressure is found to be proportional to the pressure. Let, therefore, $\rho + \frac{p}{\mu}$ be the density of the liquid under the pressure p.

Then $\kappa\left(\rho + \frac{p}{\mu}\right) d_x y.\delta x =$ mass of $PQ = \kappa\rho.\delta x$; and since the disturbance is very small, p is very small compared with $\mu\rho$, therefore $\mu\rho d_x y = \mu\rho - p$ very nearly,

$$\therefore \mu\rho d_x^2 y = -d_x p,$$

$$\therefore d_t^2 y = \mu d_x^2 y.$$

In liquids, the heat developed by compression is nearly insensible.

(3) Let AP be a rod vibrating longitudinally, $\frac{cp}{\mu\rho}$ the quantity by which the rod is shortened, or lengthened, when it is compressed longitudinally, or extended, by a pressure κp; the original length of the rod being c.

Then $d_x y.\delta x =$ length of $PQ = \left(1 - \frac{p}{\mu\rho}\right).\delta x$,

$$\therefore \mu\rho d_x^2 y = -d_x p,$$

$$\therefore d_t^2 y = \mu d_x^2 y.$$

In consequence of the variation of the thickness of a rod, when it is compressed longitudinally, or extended, the variation of its length (Pouillet Elemens de Physique, 463.) is 1,5 of what it would be if its thickness were invariable. It is the variation of the length of the rod, on the latter supposition, that is to be used in deducing the value of μ.

64. The equation of the motion of the disturbance is of the form $d_t^2 y = a^2 d_x^2 y$. The integral of this equation may be made to depend on that of $d_t z = a d_x z$, which we know to be $z = \phi(x + at)$, in the following manner:

$$d_t d_t y = a^2 d_x d_x y, \quad d_t d_x y = d_x d_t y;$$

$$\therefore\ d_t(a d_x y + d_t y) = a d_x(a d_x y + d_t y),$$

$$d_t(a d_x y - d_t y) = -a d_x(a d_x y - d_t y);$$

$$\therefore\ a d_x y + d_t y = \phi(x + at),$$

$$a d_x y - d_t y = \psi(x - at);$$

$$\therefore\ 2 a d_x y = \phi(x + at) + \psi(x - at),$$

$$2 d_t y = \phi(x + at) - \psi(x - at);$$

$$\text{and } y = F(x + at) + f(x - at).$$

65. Let the initial disturbance extend through a very small space 2α, that is from $-\alpha$ to α; then at the beginning of the motion, or when $t = 0$, $a d_x y + d_t y = \phi x$, $a d_x y - d_t y = \psi x$; and the fluid at any point distant from A by a quantity greater than α, will be at rest, therefore $d_x y = 1$, $d_t y = 0$; and therefore $\phi x = a$, $\psi x = a$, as long as x does not lie between $-\alpha$ and α. Therefore $\phi(x + at) = a$, except when $x + at$ lies between $-\alpha$ and α; and $\psi(x - at) = a$, except when $x - at$ lies between $-\alpha$ and α; hence $d_t y = 0$, except when one of the quantities $x + at$, $x - at$ lies between $-\alpha$ and α; and when $x - at$ is less than $-\alpha$, or greater than α, $x + at$ is greater than α; therefore if P, R be any two points in AP, the fluid at P will remain at rest till $AP - at = \alpha$, or till the end of the time $\frac{1}{a}(AP - \alpha)$, it will then begin to move, and will return to a state of permanent rest when $AP - at = -\alpha$, or at the end of the time $\frac{1}{a}(AP + \alpha)$. In like manner, the fluid at R will begin to move at the end of the time $\frac{1}{a}(AR - \alpha)$, and will return to a state of rest at the end of the time $\frac{1}{a}(AR + \alpha)$. Hence the fluid at R will begin

to move $\frac{1}{a}PR$ later than the fluid at P; therefore the velocity with which the disturbance is propagated

$$= PR \div \left(\frac{1}{a}PR\right) = a.$$

66. Sound is a repetition of such disturbances; and the velocity of sound in any medium, is the same as the velocity with which a small disturbance is propagated through it. When the disturbances are repeated at small and equal intervals the sound becomes a musical note.

Hence, the velocity of sound in an elastic fluid, at the temperature T, is

$$\sqrt{\{\mu(1+\text{EC})(1+\text{ET})\}}.$$

In dry air,

$$\sqrt{(\mu)} = 916{,}45 \text{ feet},\ \text{E} = 0{,}00375,\ \text{EC} = 0{,}41365\,;$$

therefore if V be the velocity of sound expressed in feet,

$$\text{V} = 1090\sqrt{\{1 + (0{,}00375)\,\text{T}\}}.$$

The density of aqueous vapour $= \frac{5}{8}$ of the density of air, under the same pressure; therefore if Υ be the tension of the vapour contained in the air at any time, Π the atmospheric pressure, μ the ratio of the pressure to the density for dry air; the density of the moist air under the pressure Π, will be equal to the density of dry air under the pressure $\Pi - \Upsilon + \frac{5}{8}\Upsilon$; and the ratio of its pressure to its density will be

$$\mu\left(1 + \frac{3}{8}\frac{\Upsilon}{\Pi}\right) \text{ nearly};$$

$$\therefore\ \text{V} = 1090\left(1 + \frac{3}{16}\frac{\Upsilon}{\Pi}\right)\sqrt{\{1 + (0{,}00375)\,\text{T}\}}.$$

When water saturated with air, at 8^0, is pressed by a column of water 33,83 feet high, its density is increased by 0,000049589 of its original density, the force of gravity being 32,18 feet;

$$\therefore\ \mu = (33{,}83)(32{,}18) \div (0{,}000049589),\ \sqrt{(\mu)} = 4686.$$

The observed velocity of sound in water at $8^0.1$ is 4708 feet.

SECTION VI.

ON RESISTANCES.

Art. 67. The resistance of a fluid on a solid moving in it, is the resultant of the excess of the pressure of the fluid on the solid in motion, above the pressure of the fluid on the solid at rest.

Let APB (fig. 32.) be a solid, moving in a fluid with the velocity V in the direction BA. Now if we communicate to the fluid and the solid a velocity V in the direction AB, the pressure of the fluid on APB will not be altered; and the solid will be at rest in a fluid moving in the direction AB with a velocity V. Hence the force with which a fluid in motion impels a solid immersed in it, is equal to the resistance of a stagnant fluid on a solid in motion, the velocity of the fluid, in one case, being equal to the velocity of the solid in the other. So also, when both the solid and fluid are in motion, the resistance on the solid, is equal to the force with which the solid at rest would be impelled by a stream moving with the relative velocity of the fluid and solid.

Let an enveloping cylinder parallel to AB touch the solid in the curve PQR. The pressure on the surface $RAPQ$, will upon the whole be greater, and the pressure upon $PBQR$, less, than when the solid and fluid are relatively at rest. In the following Articles we shall consider that part only of the resistance, which arises from the increased pressure on $RAPQ$.

It must be observed that the theory of resistances is very imperfect, and that it is useless to expect any close agreement between the results deduced from it, and those obtained by experiment.

68. To find the force with which a stream impels a plane, the plane being perpendicular to the direction of the stream.

Let P (fig. 33.) be a point in the plane; EP, the direction of the stream, perpendicular to the plane; p' the pressure at P; p the pressure of the fluid at P before the plane was immersed, or, the pressure of the fluid at the point P in a plane moving with the same velocity, and in the same direction as the fluid; ρ the density of the fluid; K the area of the plane.

Then $p' - p$ will be the resistance on an unit of the plane at P. Now (Art. 53.) $\frac{1}{2} v^2 = \int_s S - \frac{1}{\rho} p$;

and after the plane is immersed, the velocity at $P = 0$,

$$\therefore 0 = \int_s S - \frac{1}{\rho} p', \quad \therefore p' - p = \tfrac{1}{2} \rho v^2,$$

and the resistance on the plane $= (p' - p) \mathrm{K} = \frac{1}{2} \rho v^2 \mathrm{K}$.

$\frac{1}{2} \rho v^2 \mathrm{K}$ is the weight of a column of fluid having the given plane for its base, and whose altitude is the space due to the velocity of the fluid.

If the plane be made to move in a direction perpendicular to EP, it is manifest that the force with which the stream impels the plane, will not be altered. Hence the force with which a given fluid impels a given plane, depends only on that part of the relative velocity of the fluid and plane, which is perpendicular to the plane. Also, since the resistance, or impelling force of the fluid arises from the pressure of the fluid on the plane, it must act in a direction perpendicular to the plane.

69. A stream impinges obliquely on a plane; to find the force with which the stream impels the plane.

Let P (fig. 33.) be a point in the plane; AP the direction of the stream; EP perpendicular to the plane; v the velocity of the stream; R the resistance, or the force with which the stream impels the plane.

The velocity of the stream estimated in the direction EP

$$= v . \cos APE, \quad \therefore R = \tfrac{1}{2}\rho v^2 . (\cos APE)^2 \text{K}.$$

Cor. 1. The resolved part of the impelling force estimated in the direction of the stream

$$= R . \cos APE = \tfrac{1}{2}\rho v^2 . (\cos APE)^3 \text{K}.$$

Cor. 2. The resolved part of the impelling force estimated in a direction perpendicular to the stream, and in the plane APE,

$$= R . \sin APE = \tfrac{1}{2}\rho v^2 . (\cos APE)^2 . \sin APE . \text{K}.$$

70. A cylinder having the curve BPC (fig. 34.) for its base, is immersed in a stream flowing in the direction AE; to find the force with which the stream impels the cylinder in the directions AE and MA.

Draw AN perpendicular to AE; PM, QN parallel to AE; PE a normal to BP at P. Let a be the altitude of the cylinder, $MP = x$, $AM = y$, $MN = \delta y$, R the impelling force, or resistance, on that part of the cylinder which stands on BP, estimated in the direction AE, therefore ultimately $d_y R . \delta y =$ resistance on that part of the cylinder which stands on PQ

$$= \tfrac{1}{2}\rho v^2 (\cos AEP)^3 a . PQ = \tfrac{1}{2}\rho v^2 (\cos AEP)^2 a \delta y,$$

$$\text{and } \tan AEP = -d_y x, \quad \therefore d_y R = \tfrac{1}{2}\rho v^2 \frac{a}{1 + (d_y x)^2}.$$

$$\text{and } R = \tfrac{1}{2}\rho v^2 a \int_y \frac{1}{1 + (d_y x)^2}.$$

So also, if S be the resistance on the part BP of the cylinder, estimated in the direction MA,

$$d_y S . \delta y = \tfrac{1}{2}\rho v^2 (\cos AEP)^2 \sin AEP . a . PQ$$

$$= \tfrac{1}{2}\rho v^2 . \cos AEP . \sin AEP . a . \delta y;$$

$$\therefore d_y S = \tfrac{1}{2}\rho v^2 a \frac{d_y x}{1 + (d_y x)^2}, \text{ and } S = \tfrac{1}{2}\rho v^2 a \int_y \frac{d_y x}{1 + (d_y x)^2}.$$

The integrals must be taken between the limits corresponding to B and C.

71. A solid is generated by the revolution of the curve BPC (fig. 34.) round AE; to find the force with which it is impelled by a stream moving in the direction AE.

Let R be the resistance on that part of the solid which is generated by the revolution of BP round AE; then, retaining the construction and notation of the preceding Article, we have ultimately,

$$d_y R . \delta y = \tfrac{1}{2} v^2 . (\cos PEA)^3 . 2\pi . AM . PQ = \rho v^2 \pi \frac{y \delta y}{1 + (d_y x)^2};$$

$$\therefore d_y R = \pi \rho v^2 \frac{y}{1 + (d_y x)^2},$$

$$\therefore R = \rho v^2 \pi \int_y \frac{y}{1 + (d_y x)^2}, \text{ from } B \text{ to } C.$$

72. To find the resistance on a sphere.

Let the centre of the sphere be the origin of the co-ordinates, a the radius of the sphere, and, therefore, $x^2 + y^2 = a^2$ the equation to its generating circle. Then, $y + x d_y x = 0$,

$$a^2 = x^2 + y^2 = x^2 \{1 + (d_y x)^2\} = (a^2 - y^2) \{1 + (d_y x)^2\},$$

$$\therefore \frac{y}{1 + (d_y x)^2} = y - \frac{y^3}{a^2}, \quad \int_y \frac{y}{1 + (d_y x)^2} = \frac{y^2}{2} - \frac{y^4}{4a^2} + c,$$

$$(\int_{y=a} - \int_{y=0}) \frac{y}{1 + (d_y x)^2} = \frac{a^2}{4},$$

and the resistance on the sphere $= \frac{1}{4} \rho v^2 \pi a^2$.

Cor. 1. The resistance on a circular plate, the radius of which is a, $= \frac{1}{2} \rho v^2 \pi a^2$, therefore the resistance on a sphere is half the resistance on a circular plate of the same radius as the sphere.

Cor. 2. If the density of the sphere $= \sigma$, its mass $= \sigma \frac{4}{3} \pi a^3$, and the retarding force arising from the resistance of the fluid $=$ (resistance) $\div$ (mass of the sphere) $= \frac{3}{16} \frac{\rho}{\sigma} \frac{v^2}{a}$.

SECTION VII.

DESCRIPTION OF INSTRUMENTS. METHODS OF FINDING SPECIFIC GRAVITIES, &c.

Art. 73. Almost all bodies expand by heat, and contract by cold. This property furnishes the only known mode of comparing and recording the temperatures to which any body is exposed. The expansions of mercury, or air, combined with that of the glass vessel in which they are contained, are usually employed for this purpose.

74. The common mercurial thermometer is a glass tube of uniform bore, having a bulb at one end, which, with part of the tube, is filled with mercury; the other end is usually sealed, the space between it and the mercury being a vacuum.

To fill the thermometer with mercury, the air must be partly expelled from the bulb by holding it over the flame of a lamp, and then, the other end, which is open, immersed in mercury. As the bulb cools, the mercury will be forced into it by the pressure of the atmosphere. If a paper funnel be now tied round the open end, and filled with mercury; and the mercury in the bulb be heated till it boils, the remainder of the air will be driven out, and its place supplied by mercurial vapour: this condenses on cooling, and the mercury will descend from the funnel and fill the instrument completely. When it has cooled down nearly to the highest temperature intended to be measured by it, the open end must be sealed; and as it continues to cool, the mercury will descend leaving a vacuum in the upper part of the tube.

75. To graduate a thermometer.

Let the bulb, and that part of the tube which is occupied by the mercury, be immersed in melting snow, and make a mark on the tube, opposite to the extremity of the column of mercury, when it is stationary. This is the freezing point. Next let the thermometer be surrounded by the vapour of boiling water, and make a mark on the tube at the place where the extremity of the column of mercury rests, when it is stationary. This is the boiling point. The space between the freezing and boiling points is, in the centigrade thermometer, divided into 100 equal parts, called degrees; the freezing point being called 0^0, and the boiling point 100^0.

In Fahrenheit's thermometer the space is divided into 180 parts. The freezing point is marked 32^0; and the boiling point 212^0.

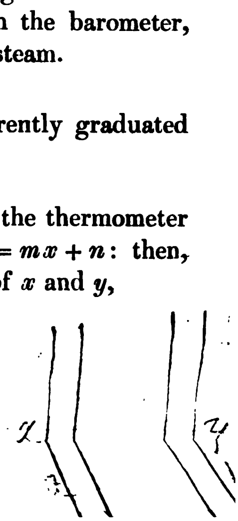
In Reaumur's thermometer the freezing point is marked 0^0, and the boiling point 80^0.

76. The temperature of melting snow is found to be the same under all circumstances. The temperature of steam, however, varies with the atmospheric pressure. 100^0 of the centigrade thermometer denotes the temperature of steam, when the pressure of the atmosphere is equal to that of a column of mercury at 0^0, 0,76 metres, or 29,9218 inches high. A variation of 1,045 inches in the height of the mercury in the barometer, occasions a change of 1^0 in the temperature of steam.

77. To compare the scales of two differently graduated thermometers.

Let x^0 of the thermometer (A), and y^0 of the thermometer (B), denote the same temperature; and let $y = mx + n$: then, if a, b; a', b' be known corresponding values of x and y,

$$\frac{y-b}{b'-b} = \frac{x-a}{a'-a}.$$



Let C^0 of the centigrade thermometer, F^0 of Fahrenheit's, R^0 of Reaumur's, denote the same temperature; then if we suppose the temperature indicated by the boiling point to be the same in all three,

$$\frac{1}{5} C = \frac{1}{9} (F - 32) = \frac{1}{4} R.$$

78. The apparent expansion of mercury in glass, between 0^0 and 100^0, is $\frac{10}{648}$ of its volume at 0^0, the difference between this and $\frac{10}{555}$, the actual expansion of mercury between the same temperatures, arises from the expansion of the glass.

79. When a solid expands by heat, its cubic expansion is three times its linear expansion.

Let the cubic and linear expansions of the solid for one degree of heat, be E, e respectively; and let V be the volume of the solid, a the distance between two given points in it, at 0^0; then, at the temperature T^0, the volume of the solid will be $V(1 + \mathrm{ET})$, and the distance between the two given points

$a(1 + e\mathrm{T})$. But $V(1 + \mathrm{ET}) : V = \{a(1 + e\mathrm{T})\}^3 : a^3$,

$\therefore 1 + \mathrm{ET} = 1 + 3e\mathrm{T}$ nearly, since $e\mathrm{T}$ is very small, $\therefore \mathrm{E} = 3e$.

Bramah's Press. (Fig. 35).

80. AB, CDE are two strong hollow cylinders communicating with each other by means of a pipe BD; M, Q two solid cylinders working in water-tight collars at A and C. The cylinder M, the diameter of which is much larger than that of Q, supports a platform F, on which the substance to be pressed is placed. Q is capable of being moved up and down by means of a lever HL having its fulcrum at H. D is a valve opening upwards; B a valve opening into the cylinder AB; E a cistern filled with water; I a cross beam, the ends of which are fastened to the upright posts G, H.

Suppose Q to be in its lowest position, and the space between the solid and hollow cylinders to be filled with water;

then on elevating Q, the atmospheric pressure forces the water in E through ED into the space previously occupied by Q; on depressing Q, the valve D closes, and a portion of the water in CD is forced through the valve B, which prevents the return of the water, into AB, and causes M to ascend. This process is repeated till the substance between F and I is sufficiently compressed. The pressure may at any time be removed by unscrewing a plug at N, which permits the water in AB to escape.

Let r, s be the radii of the cylinders M, Q; W the whole pressure on the platform F; P the force applied at L; p the pressure of the water in the cylinders.

Then W = pressure on the lower end of $M = \pi r^2 p$,

$P\dfrac{HL}{HK}$ = pressure on the lower end of $Q = \pi s^2 p$,

$$\therefore \frac{W}{P} = \frac{HL}{HK}\,\frac{r^2}{s^2}.$$

When this machine is employed to produce tension, as, for instance, in extracting piles, or in proving cables, &c. a cylindrical bar of iron passing through a water-tight collar in the bottom of the hollow cylinder AB, has one end fastened into the cylinder M, while a ring at the other end serves to connect it with the pile or cable.

The Diving Bell. (Fig. 36).

81. The diving bell is a chest, the weight of which is greater than that of the water it would contain, suspended by a rope with its mouth downwards. If the bell be lowered out of air into water in this position, the air contained in it will prevent the water from rising in the upper part of the bell, and thus enable persons to breathe at considerable depths below the surface of the water.

82. To find the space occupied by the air in the bell at any depth below the surface.

Let BE be the bell, draw AM vertical meeting the surface of the water on the outside of the bell in A, and the surface of the water within the bell in M; and let h be the altitude of a column of water whose pressure is equal to that of the atmosphere (about 34 feet). When the bell was at the surface the air in it occupied the space $DECB$, under a pressure equal to that of a column of water the height of which is h; and the pressure at M is that of a column of water whose height is $h + AM$;

$$\therefore \text{ (Art. 30.) } \frac{\text{vol. } BMC}{\text{vol. } DECB} = \frac{\text{atmospheric pressure}}{\text{pressure at } M} = \frac{h}{h + AM}.$$

The water may be almost wholly expelled from the interior of the bell by a supply of air from above, forced by an air-pump through a flexible tube terminating under the mouth of the bell. In this manner also the air is changed as often as it becomes unfit for respiration.

The Syphon. (Fig. 37.)

83. The syphon is a bent tube ABC open at both ends. Let the ends be closed, after filling it with fluid, and place it with one end in a bowl of the fluid with which it was filled, so that the other end may be below the surface of the fluid in the bowl. Let the plane of the surface of the fluid in the bowl meet the legs of the syphon in A, K; and let Π be the atmospheric pressure. Then if the end A be opened, the pressure within the tube at H will be Π; and if the end C be opened the pressure at C will be Π, and therefore the initial pressure within the tube at K will be less than Π. And since the pressure at K is less than the pressure at H, the column of fluid HBK will move in the direction HBK, and run out at C, while the fluid in the bowl is forced up AB by the pressure of the atmosphere. And this will continue till the surface descends to the level of the highest end of the syphon.

The syphon will not act when the altitude of the highest part of it above the surface of the fluid in the bowl, is greater than the height of a column of the fluid, whose pressure is equal to that of the atmosphere. For on opening A, the fluid in BA

will sink till its altitude is such that the pressure it exerts at H, becomes equal to the pressure of the atmosphere, leaving a vacuum at B.

The Common Pump. (Fig. 38.)

84. AB, BC are two hollow cylinders having a common axis; C the surface of the water into which the extremity of BC descends; M a piston capable of being moved up and down by a rod MA, and containing a valve opening upwards; AB the range of the piston; B a valve opening upwards; D a spout placed a little above A.

Suppose M to be at B ~~A~~, and the pump to be filled with air, the pressure of which is equal to that of the atmosphere; and let M be elevated to A. Then, the air in BC will open the valve B and fill AB, and the pressure of the air in the pump being less when it occupies the space ABC, than when it occupied the space BC, the pressure of the atmosphere will force the water up BC till the pressure at C is the same as before, or equal to the pressure of the atmosphere. As soon as M begins to descend, the valve B closes, and the air between M and B escapes through the valve M. The water will ascend in the pump each time this process is repeated, and will finally pass through the valves B and M; and then, when M ascends to A, it will flow through D.

If h be the altitude of a column of water whose pressure is equal to that of the atmosphere, BC must always be less than h, otherwise the water would never reach B.

Cor. If P be the surface of the water in BC, r the radius of the cylinder AB, ρ the density of water, and if we suppose M to ascend very slowly; the pressure of the air in MP $= g\rho(h - PC)$, therefore the pressure upwards on $M = g\rho\Pi r^2$ (h - $\cancel{(a - PC}$), and the pressure of the atmosphere downwards on $M = g\rho\pi r^2 h$, therefore the tension of the rod $AM = g\rho\pi r^2 . PC$.

85. To find the height through which the water rises each time the piston ascends.

Let P be the surface of the water in BC when M is at B; Q the surface of the water when M is at A. Then, the pressure of the air in $BP = g\rho\,(h - PC)$, and the pressure of the air in $AQ = g\rho\,(h - QC)$; but (pressure of the air in BP): (pressure of the air in AQ) = (vol. AQ) : (vol. BP),

$$\therefore\ h - PC : h - QC = (\text{vol.}\,AQ) : (\text{vol.}\,BP).$$

86. When AE is the range of the piston, the pressure of the air between B and M, when M is at E, must be greater than the pressure of the atmosphere, otherwise the air will not escape through the valve in M, and M will reascend without increasing the elevation of the water in BC.

Let P be the surface of the water in BC when M is at A, then, the pressure of the air in $AP = g\rho\,(h - PC)$, and when M comes to E the pressure of the air in $BE = g\rho\,(h - PC)\,AB \div EB$, and this must be greater than $g\rho h$, the atmospheric pressure, therefore $AE.h$ must be greater than $AB.PC$, and BC is the greatest value of PC, therefore $AE.h$ must be greater than $AB.BC$.

87. Suppose the whole pump to be part of the same cylinder, and the valve to be at, or near the surface of the water. Let AE (fig. 39.) be the range of the piston, P the surface of the water within the pump, C the surface of the water on the outside. When the piston is at A, the pressure of the air in $AP = g\rho\,(h - PC)$; when the piston descends to E, the pressure of the air in $EP = g\rho\,(h - PC)\,AP \div EP$, and this must be greater than $g\rho h$, the atmospheric pressure, in order that the valve in the piston may open, therefore $h.AE$ must be greater than $AP.PC$, and the greatest value of $AP.PC$ is $\frac{1}{4}\,AC^2$, therefore $4h.AE$ must be greater that AC^2.

The Forcing Pump. (Fig. 40.)

88. M is a solid piston working in a hollow cylinder ABC,
A the lower end of which is immersed in water; DF a tube
B ascending from AB; B, D valves opening upwards; AE the range of the piston.

$h : h - x ::$ volume AQ : vol BC

$:: a + b - x : b$

$\therefore x^2 - (a+b+h)\,x + h(a+b) - hb$

$\sqrt{(a+b+h)^2 + ah}.$

Let M be at E, and the pressure of the air in the pump equal to the atmospheric pressure. Let M be elevated to A, then the pressure of the air below M is diminished, and the pressure of the atmosphere will force the water up the tube BC. When M descends the valve B closes, D opens, and a portion of the air between M and B escapes through DE. When M ascends, the water rises in BC as before, and at last rises above B, and is forced up the tube DE when M descends. On elevating M, D closes, and a fresh portion of water enters AE through B, and is forced up DE by the next descent of M.

A solid cylinder working in a water-tight collar at A, is frequently used instead of the piston M.

The stream of water may be rendered continuous by means of a close vessel DF (fig. 41.) filled with air; HF is the lower extremity of the ascending tube. When the surface of the water in DF rises above H, the pressure of the air, which is condensed in the upper part of DF forces the water up HF in a continued stream.

The Fire Engine. (Fig. 42.)

89. AB, $A'B'$, are two forcing pumps, having a common air vessel DF, and suction tube C. The pistons are worked by a lever LGL', so that one descends while the other ascends. The jet of water may be pointed in any direction by means of the flexible tube F. The action of the engine is in all respects the same as that of the forcing pump.

The Condenser. (Fig. 43.)

90. AB is a hollow cylinder, of which the end B is screwed into the neck of a strong vessel C; M a piston containing a valve opening downwards; B a valve also opening downwards.

Suppose M to be at A, and the barrel AB and the receiver C to be filled with air of the same density as the atmospheric air. When M begins to descend the pressure of the air in MB, which is increased in consequence of the diminution of its volume, closes the valve M, and opens the valve B; and when M is thrust down to B, a quantity of air, which, under the pressure of the atmosphere, occupied the space AB, is forced into C; when M begins to ascend, the pressure of the air in C closes the

valve B, and the pressure of the atmosphere opens M, and when M comes to A, AB is filled with air under the pressure of the atmosphere, and this is forced into C by the next descent of M.

91. To find the density of the air in the receiver after n descents of the piston.

Let A, B, be the capacities of the receiver and barrel respectively; ρ the density of atmospheric air. Then ρ A will be the mass of the air contained in the receiver at first, and ρ B the mass of the air forced into the receiver at each descent of the piston, therefore $\rho\text{A} + n\rho\text{B}$ will be the mass of the air in the receiver after n descents of the piston; and its volume is A, therefore its density will be $\rho\left(1 + n\frac{\text{B}}{\text{A}}\right)$.

92. The gauge of a condenser is a glass tube AB (fig. 44.) sealed at A and communicating with the receiver of the condenser at B, the part AP of the tube is filled with air which is separated from the air in PB by a drop of mercury P. When the air in the receiver is condensed, P is forccd towards A, till the pressures, and, therefore, the densities of the air in AP, PB are equal. Let ρ be the density of atmospheric air; then, when the drop of mercury is at M, the density of the air in AM or MB $= \rho\frac{AP}{AM}$ (Art. 30.)

Hawksbee's Air Pump. (Fig. 45.)

93. AB, $A'B'$ are two hollow cylinders communicating at B, B', with a strong vessel by means of a pipe C; M, M' pistons containing valves opening upwards, and worked by a toothed wheel E; B, B', valves opening upwards.

Suppose M to be at A, and M' at B', and the density of the air in the receiver C, and in AB, to be equal to the density of atmospheric air. Then if E be turned so that M may descend and M' ascend, the valve B' opens, B and M' close, and a quantity of air, which at first occupied the space AB, is forced through the valve M, by the time M reaches B; when the wheel is turned in the opposite direction, the valve B opens,

M and B' close, and a quantity of air, which after the first turn of the wheel occupied the space $A'B'$, is forced through M' by the descent of M' from A' to B'. The exhaustion may be carried on to any required extent, by a repetition of this process.

94. To find the density of the air in the receiver after any number of turns of the wheel E.

Let A, B, be the capacities of the receiver and barrel respectively; ρ the density of the air in the machine, ρ_1, $\rho_2 \ldots \rho_n$ the densities of the air after 1, 2,......n turns. Then, the air, which occupied the space A when M was at B, will occupy the space A + B when M comes to A, therefore $\rho_1 (A + B) = \rho A$, similarly $\rho_2 (A + B) = \rho_1 A$, and so on,

$$\therefore \rho_n (A + B)^n = \rho A^n.$$

Cor. Hence if h be the altitude of the mercury in a barometer, σ the density of mercury, and therefore $g\sigma h$ the pressure of the atmosphere, the pressure of the air in the receiver after n turns will be $g\sigma h \left(\frac{A}{A + B}\right)^n$.

The employment of two pistons worked by the same wheel diminishes considerably the labour of working the pump. For the pressures of the atmosphere on the upper surfaces of M, M' being equal, the pump may be worked by a force sufficient to overcome the friction together with the difference of the pressures on the under surfaces of M, M'; while the ascent of a single piston is opposed by the friction together with the difference between the pressures on its upper and under surfaces.

Smeaton's Air Pump. (Fig. 46.).

95. AB is a hollow cylinder communicating with the receiver by means of the pipe BC; M a piston worked by a rod AM passing through an air tight collar in a plate which closes the upper end of the cylinder; at A, M, B, are placed valves opening upwards.

Let A, B be the capacities of the receiver and barrel respectively; ρ the density of the air in the machine; and suppose M to be at A. Then, as soon as M begins to

descend the valves A and B will close, and the valve at M will open: when M reascends from B to A, the valves A and B will open, and the valve at M will close; and the air which occupied the space A before M left A, will occupy the space A + B when M returns to A.

Hence, if ρ_1, $\rho_2 \ldots\ldots \rho_n$ be the densities of the air in the receiver after 1_1 $2_1 \ldots\ldots n$ descents and ascents of the piston, $\rho_1(\mathrm{A}+\mathrm{B}) = \rho\mathrm{A}$, similarly $\rho_2(\mathrm{A}+\mathrm{B}) = \rho_1\mathrm{A}$, and so on,

$$\therefore\ \rho_n(\mathrm{A}+\mathrm{B})^n = \rho\,\mathrm{A}^n.$$

The valve A, which closes as soon as M begins to descend, relieves M from the pressure of the atmosphere, and the valve in M is opened by a very small pressure of the air beneath. On this account Smeaton's pump is capable of producing a greater degree of exhaustion than Hawksbee's. Also the removal of the pressure of the atmosphere on M, diminishes the labour of working this pump.

96. The receiver is usually a strong glass jar, having its mouth ground truly plane, placed with its mouth downwards on a plane surface of brass, into which the extremity of the tube C is inserted. The junction of the receiver and the plate of brass is rendered impervious to the air by smearing the edge of the receiver with some unctuous substance.

The valves are formed of a triangular piece of oiled silk stretched over a grated orifice in a plate of metal, to which the corners of the triangle are fastened. When the air presses on the upper surface of the valve, the silk is brought into contact with the edge of the orifice, and the passage of the air through it is prevented. When the air presses on the under side of the valve, the silk is lifted up from the grating, and the air finds a free passage between the silk and the plate of the valve.

97. The barometer gauge is a vertical glass tube not less than 31 inches long, the lower end of which is immersed in a cistern of mercury, while its upper end communicates with the receiver.

If z be the altitude of the mercury in the gauge above the surface of the mercury in the cistern, the pressure of the air in the receiver $= g\sigma h - g\sigma z$. Hence, the density of the air in the receiver $= \rho \dfrac{h - z}{h}$.

98. The syphon gauge is a glass tube $ABCD$ (fig. 47.) closed at A, and communicating with the receiver at D; AB and part of BC is filled with mercury. As the exhaustion proceeds, the mercury sinks in AB and rises in BC; and if z be the perpendicular distance between the surfaces of the mercury in AB and BC, $g\sigma z$ will be the pressure at the surface of the mercury in BC, or the pressure of the air in the receiver.

THE COMMON BAROMETER. (Fig. 48.)

99. The principle of this instrument is explained in Art. 28. In order to avoid the trouble of observing the altitudes of both extremities of the column of mercury, the diameter of the tube BC is made much greater than that of the tube AB, and a scale of inches is attached to AB. Let zero of the scale of inches be at K; and when the plane of the surface of the mercury in BC passes through K, let H be the upper extremity of the column of mercury. And when the extremity of the mercurial column is at P, let the plane of the surface of the mercury in BC pass through Q; and let H, K be the areas of horizontal sections of the tubes AB, BC, respectively. Then, $\mathrm{H}.HP = \mathrm{K}.KQ$, and HP is the apparent variation of the altitude of the mercury, but its real variation

$$= HK - PQ = HP + KQ = \left(1 + \frac{\mathrm{H}}{\mathrm{K}}\right).HP,$$

and the true altitude of the mercury $= PQ = HK - \left(1 + \dfrac{\mathrm{H}}{\mathrm{K}}\right).HP$.

In some barometers the cistern is constructed as in fig. 49, and the mercury in it is elevated or depressed by a screw, till its surface touches a fine point of ivory, which is in the same horizontal plane with the zero of the scale when the tube AB is vertical.

In order to obtain the true height of a column of mercury, whose pressure is equal to that of the atmosphere, we must add the capillary depression of the mercury in *AB* to the observed altitude. The exact amount of the depression in glass tubes of different diameters appears to be rather uncertain. Hence, in order to determine accurately the absolute height of the mercury, the observations must be made with a barometer having a tube of such large internal diameter that the depression may be nearly insensible; or with the syphon barometer (fig. 17.) in which, on account of the equality of the tubes *AB*, *BC*, the extremities of the columns of mercury in them are equally depressed.

100. To compare the specific gravities of air and water.

Let a large glass flask capable of being closed by a stop-cock, be exhausted as completely as possible, and weighed. Permit the air to enter the flask and weigh it again. Lastly weigh the flask when filled with water.

Let X be the weight of the exhausted flask, Y its weight when filled with air, W its weight when filled with water. Then $Y - X$ is the weight of the air contained in the flask, $W - X$ the weight of the water contained in it. Therefore, since $Y - X$, $W - X$, are the weights of equal volumes of air and water respectively, (Art. 10.) (specific gravity of air) : (specific gravity of water) $= Y - X : W - X$.

According to Biot, the specific gravity of dry air at 0^0, under the pressure of 29,9218 inches of mercury in lat. 45^0, $= (0{,}0012991) \cdot$(specific gravity of water), the water being at 4^0, the temperature at which its density is a maximum.

101. To determine the weight of a given volume of water.

Let a sphere, cube, or cylinder, of known dimensions, be weighed in air and in water. Let V be the volume of the sphere; W its apparent weight in air; X its apparent weight when suspended in water; U the weight of the air displaced by it; W', X' the weights of the air displaced by the weights W, X. Then, weight of the sphere − weight of the air displaced by it $= W - W'$; weight of the sphere − weight of

the water displaced by it $= X - X'$; therefore, the weight of the water displaced by the sphere, or the weight of a volume V of water $= W - X + U - W' + X'$.

A brass sphere appeared to weigh 28704,5 grains when suspended in air, and 49,8 grains when suspended in water; the volume of the sphere at $16^0\frac{2}{3}$ was 113,5264 cubic inches; the temperature of the water $18^0,9$; the temperature of the air $19^0,44$; the altitude of the mercury in the barometer 29,74 inches; the weights were of brass, the density of which is probably about $8\frac{1}{2}$ times the density of water. Brass expands ,0000567 of its volume for one degree of heat, therefore the volume of the sphere at $18^0,9$, = 113,5407 cubic inches. $W - X = 28654,7$ grains, therefore, neglecting the small quantities U, W', X', the weight of a cubic inch of water at $18^0,9 = (W - X) \div V = 252\frac{1}{2}$ grains nearly, and water expands 0,001543 of its volume between 4^0 and $18^0,9$, therefore the weight of a cubic inch of water at $4^0 = 253$ grains nearly. U = weight of 113,54 cubic inches of air at $19^0,44$, under the pressure of 29,74 inches of mercury at $19^0,44$ =

$$\{(113{,}54)\,(253)\,(0{,}0013)\,.\,(29{,}74)\,[1 - (0{,}00018)\,(19{,}44)]\}$$
$$\div \{[1 + (0{,}00375)\,(19.44)]\,(29{,}92)\} = 34{,}47 \text{ grains.}$$

$W' - X'$ = weight of air displaced by 28654,7 grains of brass $= \frac{2}{17}(34,47) = 4,05$ grains. Hence, the weight of 113,5407 cubic inches of water at $18^0,9 = 28685,12$ grains, and the weight of one cubic inch = 252,642 grains.

The weight of a cubic inch of water at its maximum density $= (252,642).(1,001543) = 253,032$ grains.

The weight of a cubic inch of water at $16^0\frac{2}{3}$ (62^0 *F*) = 252,746 grains.

102. To compare the specific gravities of a solid and a fluid, by weighing the solid in air and in the fluid.

Let the solid be suspended by a fine wire, or a hair, from the pan of a balance as in (fig. 50.), and let W be the weight of the solid in air, X its apparent weight when suspended in the fluid. Then, (Art. 22.) neglecting the weight of the air displaced by the solid,

$$\text{weight of the solid} - \text{weight of the fluid displaced} = \text{X};$$

$$\text{therefore, weight of the fluid displaced} = \text{W} - \text{X};$$

and since W and W − X are the weights of equal volumes of the solid and of the fluid,

$$S.G.\text{ solid} : S.G.\text{ fluid} = \text{W} : \text{W} - \text{X}.$$

When great accuracy is required, the weight of the air displaced by the solid, must be taken into account.

Let W be the weight of the solid in air, U the weight of the air displaced by it; X the apparent weight of the solid when suspended in water. Then,

$$\text{weight of the solid} - \text{U} = \text{W};$$

$$\text{weight of the solid} - \text{weight of the fluid displaced by it} = \text{X};$$

$$\text{therefore the weight of the solid} = \text{W} + \text{U},$$

$$\text{and the weight of the fluid displaced by it} = \text{W} - \text{X} + \text{U};$$

$$\text{therefore } S.G.\text{ solid} : S.G.\text{ fluid} = \text{W} + \text{U} : \text{W} - \text{X} + \text{U}.$$

103. When the weight of the solid is less than the weight of the fluid displaced by it, it must be fastened to another solid of sufficient density and magnitude to cause both to sink. Let X = apparent weight of the denser solid in the fluid − apparent weight of both solids in the fluid = weight of the fluid displaced by the rarer solid − weight of the rarer solid;

$$\text{therefore the weight of the fluid displaced} = \text{W} + \text{X};$$

$$\text{therefore } S.G.\text{ solid} : S.G.\text{ fluid} = \text{W} : \text{W} + \text{X}.$$

If we take into account the weight of the air displaced,

$$\text{weight of the solid} - \text{weight of air displaced by it} = \text{W};$$

$$\text{weight of the fluid displaced by the solid} - \text{weight of the solid} = \text{X};$$

$$\text{therefore weight of the fluid displaced} = \text{W} + \text{U} + \text{X};$$

$$\text{therefore } S.G.\text{ solid} : S.G.\text{ fluid} = \text{W} + \text{U} : \text{W} + \text{U} + \text{X}.$$

104. To compare the specific gravities of two fluids by weighing the same solid in each.

Let W be the weight of the solid in air, X its apparent weight when suspended in the fluid (A), Y its apparent weight when suspended in the fluid (B). Then, neglecting the weight of the air displaced by the solid.

weight of the solid − weight of the fluid (A) displaced by it = X;

weight of the solid − weight of the fluid (B) displaced by it = Y;

therefore weight of the fluid (A) displaced = W − X,

and weight of the fluid (B) displaced = W − Y;

and W − X, W − Y are the weights of equal volumes of the fluids (A) and (B) respectively; therefore

$$S.G.\text{fluid }(A) : S.G.\text{fluid }(B) = W - X : W - Y.$$

If the weight of the air displaced by the solid = U

weight of the fluid (A) displaced = W + U − X;

weight of the fluid (B) displaced = W + U − Y;

therefore $S.G.$fluid (A) : $S.G.$fluid (B)

$$= W + U - X : W + U - Y.$$

105. To compare the specific gravities of two fluids (A) and (B) by weighing equal volumes of each.

Let X be the weight of a flask filled with the fluid (A) and closed with a ground stopper; Y the weight of the flask similarly filled with the fluid (B). W the weight of the flask. Then, neglecting the weight of the air contained in the flask,

weight of the fluid (A) contained in the flask = X − W;

weight of the fluid (B) contained in the flask = Y − W;

and X − W, Y − W are the weights of equal volumes of the fluids (A) and (B) respectively,

$$\therefore S.G.\text{fluid }(A) : S.G.\text{fluid }(B) = X - W : Y - W.$$

If the weight of the air contained in the flask $= \text{U}$; then,

weight of the fluid (A) contained in the flask $= \text{X} - \text{W} + \text{U}$;

weight of the fluid (B) contained in the flask $= \text{Y} - \text{W} + \text{U}$;

$\therefore$ $S.G$ fluid (A) : $S.G$ fluid (B) $= \text{X} - \text{W} + \text{U} : \text{Y} - \text{W} + \text{U}$.

106. The specific gravity of a solid broken into small fragments may be found in the following manner.

Let W be the weight of the solid in air; X the weight of a flask filled with the fluid; Y the weight of the flask containing the fragments of the solid, and filled up with the fluid. Then, neglecting the weight of the air displaced by the solid,

weight of the solid − weight of the fluid displaced by it $= \text{Y} - \text{X}$;

$\therefore$ weight of the fluid displaced $= \text{W} - \text{Y} + \text{X}$;

$\therefore$ $S.G.$solid : $S.G.$fluid $= \text{W} : \text{W} - \text{Y} + \text{X}$.

If the weight of the air displaced by the solid $= \text{U}$,

$S.G.$solid : $S.G.$fluid $= \text{W} + \text{U} : \text{W} - \text{Y} + \text{X} + \text{U}$.

The Common Hydrometer. (Fig. 51.)

107. E, D are two hollow spheres having their centres in the axis of a graduated cylindrical stem EC. D is loaded with lead so that the centre of gravity of the whole instrument may be below the centre of gravity of the fluid displaced by the spheres E, D. The instrument is used in comparing the specific gravities of fluids.

Let W be the weight of CED, V its volume; K the area of a section of the stem EC. When it floats vertically in a fluid (A), let the surface of the fluid meet EC in P; and when it floats vertically in a fluid (B), let the surface of the fluid meet EC in Q. Then, since $\text{V} - \text{K}.CP$, and $\text{V} - \text{K}.CQ$ are the volumes of the fluids (A) and (B) displaced by the instrument, and the weight of a floating solid is equal to the weight of the fluid displaced by it (Art. 22. Cor. 2.),

$$\text{W} = \{S.G.\text{fluid } (A)\}(\text{V} - \text{K}.CP);$$

$$\text{W} = \{S.G.\text{fluid } (B)\}(\text{V} - \text{K}.CQ);$$

$\therefore$ $S.G.$fluid (A) : $S.G.$fluid $(B) = \text{V} - \text{K}.CQ$: $\text{V} - \text{K}.CP$.

SIKES' HYDROMETER. (Fig. 82.)

108. This instrument differs from the preceding in the form of the stem EC, which is a very thin flat bar, and in having a series of weights capable of being fixed on the stem connecting E and D, of such magnitude that when DC floats with nearly the whole of it stem above the surface of the fluid, the addition of one of the weights causes it to sink nearly to C.

Let V be the volume of the instrument, W its weight; K the area of a section of the stem EC. When it floats in a fluid (A), let X be the weight at C, P the surface of the fluid; when it floats in a fluid (B), let Y be the weight at C, Q the surface of the fluid; and let R, S be the volumes of the weights X, Y. Then,

the weight of the fluid (A) displaced $= \text{W} + \text{X}$;

the weight of the fluid (B) displaced $= \text{W} + \text{Y}$;

the volume of the fluid (A) displaced $= \text{V} + \text{R} - \text{K}.CP$;

the volume of the fluid (B) displaced $= \text{V} + \text{S} - \text{K}.CQ$;

$$\therefore \text{W} + \text{X} = \{S.G.\text{fluid } (A)\}(\text{V} + \text{R} - \text{K}.CP);$$

$$\text{W} + \text{Y} = \{S.G.\text{fluid } (B)\}(\text{V} + \text{S} - \text{K}.CQ);$$

$$\therefore S.G.\text{fluid } (A) : S.G.\text{fluid } (B)$$

$$= (\text{W} + \text{X})(\text{V} + \text{S} - \text{K}.AQ) : (\text{W} + \text{Y})(\text{V} + \text{R} - \text{K}.AP).$$

NICHOLSON'S HYDROMETER. (Fig. 53.)

109. EF is a hollow cylinder of copper; C a dish supported by a slender steel wire CE placed in the axis of EF; D a heavy dish suspended from the lower extremity of EF. This instrument is used in comparing either the specific gravity of a fluid with that of a solid, or the specific gravities of two fluids with each other.

(1). To compare the specific gravities of a solid and a fluid.

Let Z be the weight, which placed in C, causes the instrument to sink in the fluid till the surface of the fluid meets EC in a given point H. Place the solid in C and let X be the weight which must be added, to make the instrument sink to H. Place the solid in D, and let Y be the weight which must be placed in C in order to sink the instrument to H. Then, neglecting the weight of the air displaced by the solid,

$$\text{weight of the solid} = Z - X;$$

weight of the solid − weight of the fluid displaced = apparent weight of the solid in the fluid = $Z - Y$;

$$\therefore \text{ weight of the fluid displaced} = Y - X;$$

$$\therefore\ S.G.\text{solid} : S.G.\text{fluid} = Z - X : Y - X.$$

If the weight of the air displaced by the solid = U,

$$\text{weight of the solid} = Z - X + U;$$

$$\therefore \text{ weight of the fluid displaced} = Y - X + U;$$

$$\therefore\ S.G.\text{solid} : S.G.\text{fluid} = Z - X + U : Y - X + U.$$

(2). To compare the specific gravities of two fluids (A) and (B).

Let W be the weight of the hydrometer, X the weight which must be placed in C to sink the instrument to H in the fluid (A); Y the weight which must be placed in C to sink the instrument to H in the fluid (B). Then,

$$\text{weight of the fluid } (A) \text{ displaced} = W + X;$$

$$\text{weight of the fluid } (B) \text{ displaced} = W + Y;$$

and the volume of the fluid displaced is the same in both cases,

$$\therefore\ S.G.\text{fluid } (A) : S.G.\text{fluid } (B) = W + X : W + Y.$$

Meikle's, or Hare's Hydrometer. (Fig. 54.)

110. CD, EF are two vertical glass tubes communicating at E and F with a cavity G, which is connected with some contrivance for partially exhausting the air contained in it.

C and E are immersed in cups containg two fluids (A) and (B), whose specific gravities are to be compared. If the air in G be now partially exhausted the fluids will ascend in the tubes. Let the fluid (A) ascend to P, and the fluid (B) to Q; and let the surfaces of the fluids (A) (B) in the cups meet the tubes CD, EF in C, E. Then, if the atmospheric pressure $= \Pi$, and the pressure of the air in $G = \mathrm{M}$, (Art. 12.)

$$\Pi - \mathrm{M} = \{S.G.\text{fluid } (A)\}.CP;$$

$$\Pi - \mathrm{M} = \{S.G.\text{fluid } (B)\}.EQ;$$

$$\therefore\ S.G.\text{fluid } (A) : S.G.\text{fluid } (B) = EQ : CP.$$

When the tubes are small, the altitudes CP, EQ must be diminished by the spaces through which the fluids are elevated by capillary attraction. Or the effect of capillary attraction may be eliminated in the following manner. Let a, b, be the capillary elevations of the fluids in the tubes CD, EF; then,

$$\{S.G.\text{fluid } (A)\}.(CP - a) = \{S.G.\text{fluid } (B)\}(EQ - b).$$

Permit some air to enter G, and let P', Q', be the extremities of the columns of fluid in CD, EF; therefore

$$\{S.G.\text{fluid } (A)\}(CP' - a) = \{S.G.\text{fluid } (B)\}(EQ' - b);$$

$$\therefore\ \{S.G.\text{fluid } (A)\}PP' = \{S.G.\text{fluid } (B)\}QQ',$$

$$\text{and } S.G.\text{fluid } (A) : S.G.\text{fluid } (B) = QQ' : PP'.$$

Say's Instrument for Measuring the Volumes of Small Solids. (Fig. 55.)

111. PC is a glass tube of uniform bore terminating in a cup PE having its mouth ground truly plane, and capable of being closed so as to be air-tight by a plate of glass E; within PE is a cup B containing the substance whose volume is sought. Take off the plate E, and immerse PC vertically in mercury till the surface of the mercury meets the tube in a given point P; close the cup PE with the plate E, and elevate the tube PC till the surface of the mercury on the outside meets the tube in any point C; and let M be the extremity of the column of mercury within the tube.

Let u be the volume of the space occupied by the air in PE before the solid was placed in the cup B; v the volume of the solid; K the area of a section of the tube PC; h the altitude of the mercury in the barometer; σ the density of mercury. When the surface of the mercury was at P, the air in EP occupied the space $u - v$, and its pressure $= g\sigma h$; when the surface of the mercury within the tube is at M, and the surface of the mercury on the outside at C, the air in EPM occupies the space $u - v + \text{K} . PM$, and its pressure $= g\sigma(h - MC)$; therefore (Art. 30.)

$$u - v + \text{K}.PM : u - v = h : h - MC;$$

$$\therefore v = u - \frac{h - MC}{MC}\,\text{K}.PM.$$

u may be found by a similar process, the cup B being empty. K is found by weighing the mercury occupying a given portion of the tube PC. A cubic inch of mercury at 16^0 weighs $3429\frac{1}{2}$ grains nearly, therefore if the length of the column of mercury in PC expressed in inches $= a$, and its weight in grains $= w$, $w = 3429\frac{1}{2}$ (vol. mercury in PC) $= 3429\frac{1}{2}.\text{K}a$, K being expressed in square inches.

If the weight of the solid $=$ W, its specific gravity $=$ W $\div v$. In this manner the specific gravities of powders and soluble substances are found, when the other methods, which require the substance to be immersed in fluid, cannot be used.

THE PIEZOMETER. (Fig. 56.)

112. This instrument, by means of which the compressibility of liquids may be exhibited and measured, consists of a thermometer tube DC open at C, enclosed in a strong glass vessel EF. CD is nearly filled with the liquid to be examined, and then a drop of mercury is introduced to keep the liquid within the tube separate from the liquid without; EF is then filled with water, and the required pressure produced by a piston, which is pressed down by turning a screw G. The pressure is measured by means of a guage AB similar to the one described in Art. 92, and the decrement of the volume of the fluid in CD, is deduced from the space through which the drop of

mercury descends, the area of a section of the tube and the capacity of the bulb D having been found by weighing the mercury contained in the bulb and in a given length of the tube.

The apparent diminution of the volume of the fluid in CD requires a slight correction for the alteration of the capacity of D arising from the compressibility of glass.

The Hydraulic Ram. (Fig. 57).

113. AB is a pipe descending obliquely from a reservoir of water A, to an air vessel G, into which is inserted the ascending pipe FH. C is a smaller air vessel; B a large valve opening downwards; D a valve opening upwards; E a small valve opening into C.

Suppose the valves B, E, closed by the pressure of the water in AB; D closed by its own weight; G, C filled with air; and FH filled with water up to the level of the surface of the water in A. Let the valve B be depressed; then the water in AB will move in the direction AB, and flow out at B, till the valve B carried upwards by the stream closes the orifice at B; the water in AB having its motion thus suddenly checked, will exert a very great pressure on the inner surface of the pipe AB, and will rush into the air vessel G, and up the pipe FH, compressing at the same time the air in G and C. As soon as the water in AB comes to rest, D closes, and the pressure of the air in C causes the water in AB to recoil till the air in C occupies a larger space than it did under the pressure of the atmosphere; at this instant, the pressure at B being less than the pressure of the atmosphere, B descends, and the action of the machine is renewed.

In this manner the water ascends in FH at each successive impulse, till it reaches the place to which it is desired to elevate it. A portion of the air in G and C is taken up by the water, which absorbes a considerable quantity of air under a high pressure; to supply the waste arising from this cause, the machine is provided with the valve E, which opens and permits the air to enter, during the recoil of the water in AB.

The Atmospheric Steam Engine. (Fig. 58).

114. *AB* is a hollow cylinder communicating with a boiler by means of a pipe *C*; *B* a valve opening downwards and closed by a spring; *ED* a pipe leading from a cistern of cold water *E*; *M* a piston connected with one extremity of a lever *LGF*; from the other extremity of the lever is suspended *FH*, the rod by which the machinery worked by the steam-engine is put in motion. *H* is a weight equal to half the pressure of the atmosphere on the upper surface of *M*. An apparatus connected with *FL* opens the cock *C*, when *M* descends to *B*, and closes it when *M* ascends to *A*. The cock *D* is opened in the same manner when *M* comes to *A*, and is closed again soon after *M* begins to descend.

Suppose *M* to be at *B*, and the pressure of the steam in the boiler a little greater than the pressure of the atmosphere; then, when *C* is opened, the steam rushes into *MB*, and the pressures upon the upper and lower surfaces of *M* being nearly equal, the weight *H* will cause *M* to ascend. When *M* comes to *A*, *C* is closed, and *D* is opened; a jet of cold water issues into the cylinder and condenses the steam leaving a vacuum below *M*; and since the pressure of the atmosphere on *M* is equal to twice the weight of *H*, *M* will descend with a moving force equal to the weight of *H*. When *M* arrives at *B*, *C* is opened again, and *M* ascends as before.

The water remaining in *MB* escapes through the valve *B*, which is forced open by the pressure of the steam when first admitted.

Watt's Steam Engine. (Fig. 59).

115. *AB* is a hollow cylinder closed at both ends. *LGF* is a lever, one end of which is connected with the piston *M*, by a rod *AM* passing through a steam tight collar at *A*; the other end of the lever is attached to the crank of a fly wheel. *D* is a vessel, called the condenser, surrounded with cold water; *R S* a tube connecting *AB* with the boiler and with *D*. At *R* and *S* are placed valves connected with the fly wheel in such a manner that when *M*

comes to A, a communication is opened between AM and the boiler, and between MB and the condenser, and is closed again when M has described one third of AB; and when M comes to B, a communication is opened between MB and the boiler, and AM and the condenser, and closed, as in the former case, when M has described one third of BA.

Suppose M to ascend from B to A, the space below M being filled with steam from the boiler, as soon as M arrives at A, the communication is opened between MB and D, and the steam in MB flows into D, and is there condensed leaving a vacuum in MB; at the same time, a communication being open between AM and the boiler, steam flows into AM, and M is forced downwards by the full pressure of the steam, during one third of its descent, and after the communication between AM and the boiler is cut off, by the diminished pressure of the steam in the cylinder. In the same manner, when M arrives at B, a vacuum is produced in AM by the condensation of the steam, and M is pressed upwards by the steam admitted below.

The condensation of the steam in D is promoted by a jet of cold water, which is removed as fast as it collects by a pump P.

A description of the contrivances for regulating the supply of steam and water, and for making the extremity of the piston rod describe a curve approaching to a straight line; as well the enumeration of the advantages of this construction over the atmospheric engine, would be improper in this place, on account of its length.

APPENDIX.

ART. 116. THE proposition which forms the subject of (Art. 5.) may be proved by experiment, in the following manner.

Let a vessel *ABCD* (fig. 60.) closed on all sides, and exactly filled with fluid, be placed with the side *AD* horizontal, and therefore free from any pressure arising from the weight of the fluid. At *E* and *F* make two equal orifices, and apply any pressure to the surface of the fluid at *E* by means of a piston; then, in order to prevent the fluid from escaping at *F*, another piston must be applied and pressed with exactly the same force as that at *E*. Thus the pressure communicated perpendicularly downwards at *E*, has, by the intervention of the fluid, been made to act with the same force perpendicularly upwards at *F*. If the orifice *F*, instead of being made in the upper horizontal surface of the vessel, be made at any point *G* in the inclined side, and a force applied sufficient to counteract the effort made by the fluid to escape; then, if any pressure be applied at *E*, it will be found, as before, that an additional pressure equal to that at *E*, must be applied at *G* perpendicular to the side *CD*, to preserve the equilibrium. We may conclude therefore that a force impressed on a given surface in any part of a fluid, produces an equal pressure on an equal surface in any other part of the fluid.

117. To find the pressure of a fluid on any surface.

Suppose the surface (*S*) divided into an indefinite number of portions *A*, *B*, *C*, &c. so small that every point of any one of them may be considered as at the same perpendicular depth below the surface of the fluid; and let their respective perpen-

dicular depths be a, b, c, &c.; then (Art. 12.) the pressure of the fluid on any one of them $A = g\rho Aa$; ρ being the density of the fluid, similarly, the pressure on $B = g\rho\, Bb$, &c.; therefore the sum of the pressures

$$= g\rho\,(Aa + Bb + Cc + \&c.)$$

But if the depth of the centre of gravity of S below the surface of the fluid $= X$; then, since A, B, C, &c. may be considered as bodies whose perpendicular distances from the surface of the fluid are a, b, c, &c., (Wood's Mechanics, 172.)

$$Aa + Bb + Cc + \&c. = (A + B + C + \&c.)\,X = SX;$$

$$\therefore \text{ the pressure of the fluid on } S = g\rho SX.$$

A vessel of the form of a cone with its base downwards, is filled with fluid; to compare the pressure on the base of the vessel with the weight of the fluid contained in it.

Let the cone be generated by the revolution of the right-angled triangle ABC (fig. 61.) round AC, ρ the density of the fluid; then, the area of the base of the cone $= \pi . CB^2$, and the depth of its centre of gravity below the surface of the fluid $= AC$; therefore the pressure on the base of the cone $= g\rho\pi . AC . BC^2$; and the weight of the fluid in the cone $= g\rho \dfrac{\pi}{3} AC . BC^2$; therefore pressure on the base of the cone $= 3 .$ (weight of the fluid).

Since the pressure on the base of a vessel filled with a given fluid, depends only on its area, and the depth of its centre of gravity below the surface of the fluid, the pressure on the base of the cone BAB' (fig. 61) is equal to the pressure on the base of the cylinder $BAA'B'$ (fig. 62), or the pressure on the base of the truncated cone $BAA'B'$ (fig. 63); the area of the base and the depth of its centre of gravity below the surface of the fluid being the same in each case.

118. A hollow sphere is just filled with fluid; to compare the pressure on the internal surface of the sphere with the weight of the fluid.

Let a be the radius of the sphere; then, the area of the surface of the sphere $= 4\pi a^2$, and the depth of its centre of gravity below the surface of the fluid $= a$, therefore the pressure on the surface of the sphere $= g\rho 4\pi a^3$; and the weight of the fluid contained in the sphere $= g\rho \frac{4}{3}\pi a^3$; therefore the pressure on the surface of the sphere $= 3$ (weight of the fluid).

119. To find the centre of pressure of the triangle AOB (fig. 64.) having the side OB perpendicular to the surface of the fluid, and the side OA in the surface.

Draw HR parallel to AO; and let X, Y be the distances of the centre of pressure from OA, OB respectively; $OH = x$. Then, (Art. 17.)

$$X \int_x \int_y x = \int_x \int_y x^2, \quad Y \int_x \int_y x = \int_x \int_y xy;$$

the integrals being taken between the limits corresponding to the boundary of the figure.

$$\int_y x = xy + C;$$

$$\left(\int_{y=HR} - \int_{y=0}\right) x = x.HR = AO\left(x - \frac{x^2}{OB}\right);$$

$$\int_x \left(x - \frac{x^2}{OB}\right) = \frac{x^2}{2} - \frac{x^3}{3OB} + C;$$

$$\left(\int_{x=OB} - \int_{x=0}\right)\left(x - \frac{x^2}{OB}\right) = \frac{OB^2}{2} - \frac{OB^3}{3OB} = \frac{1}{6}OB^2;$$

$\therefore \int_x \int_y x$, between the proper limits, $= \frac{1}{6} AO.OB^2$.

$$\int_y x^2 = x^2 y + C;$$

$$\left(\int_{y=HR} - \int_{y=0}\right) x^2 = x^2.HR = AO\left(x^2 - \frac{x^3}{OB}\right);$$

$$\int_x \left(x^2 - \frac{x^3}{OB}\right) = \frac{x^3}{3} - \frac{x^4}{4OB} + C;$$

$$\left(\int_{x=OB} - \int_{x=0}\right)\left(x^2 - \frac{x^3}{OB}\right) = \frac{OB^3}{3} - \frac{OB^4}{4OB} = \frac{1}{12}.OB^3;$$

$$\therefore\ \int_x \int_y x^2, \text{ between the proper limits, } = \tfrac{1}{12}.AO.OB^3.$$

$$\int_y xy = xy^2 + C;$$

$$\left(\int_{y=HR} - \int_{y=0}\right) xy = x.HR^2 = AO^2.\left(x - \frac{2x^2}{OB} + \frac{x^3}{OB^2}\right);$$

$$\int_x \left(x - \frac{2x^2}{OB} + \frac{x^3}{OB^2}\right) = \frac{x^2}{3} - \frac{2x^3}{3OB} + \frac{x^4}{4OB^2} + C;$$

$$\left(\int_{x=OB} - \int_{x=0}\right)\left(x - \frac{2x^2}{OB} + \frac{x^2}{OB^2}\right)$$

$$= \frac{OB^2}{2} - \frac{2OB^3}{3OB} + \frac{OB^4}{4OB^2} = \frac{1}{12} OB^2;$$

$$\therefore\ \int_x \int_y xy, \text{ between the proper limits, } = \tfrac{1}{12} AO.OB^2.$$

$$\therefore\ X = \tfrac{1}{2} OB, \quad Y = \tfrac{1}{2} OA.$$

120. To find the centre of pressure of a semicircle ORA (fig. 65.) having its diameter OA perpendicular to the surface of the fluid, and the extremity O of the diameter in the surface of the fluid.

Let the plane of the semicircle meet the surface of the fluid in Oy. Draw HR parallel to Oy; and let $OA = 2a$, $OH = x$.

$$\int_y x = xy + C;$$

$$\left(\int_{y=HR} - \int_{y=0}\right) x = x.HR = x\sqrt{(2ax - x^2)};$$

$$\left(\int_{x=2a} - \int_{x=0}\right) x\sqrt{(2ax - x^2)} = \frac{\pi}{2}.a^3;$$

$$\therefore\ \int_x \int_y x, \text{ between the proper limits, } = \frac{\pi}{2} a^3.$$

$$\int_y x^2 = x^2 y + C$$

$$(\int_{y=HR} - \int_{y=0}) x^2 = x^2 . HR = x^2 \sqrt{(2ax - x^2)};$$

$$\int_x x^2 \sqrt{(2ax - x^2)} = C - \tfrac{1}{4} x (2ax - x^2)^{\frac{3}{2}}$$

$$- \tfrac{5}{4} - \tfrac{1}{3} a (2ax - x^2)^{\frac{3}{2}} + \tfrac{5}{4} a^2 \int_x \sqrt{2ax - x^2};$$

$$(\int_{x=2a} - \int_{x=0}) x^2 \sqrt{(2ax - x)} = \tfrac{5}{8} \pi a^4;$$

$\therefore \int_x \int_y x^2$, between the proper limits, $= \frac{5}{8} \pi a^4$.

$$\int_y xy = \tfrac{1}{2} xy^2 + C;$$

$$(\int_{y=HR} - \int_{y=0}) xy = \tfrac{1}{2} x HR^2 = \tfrac{1}{2} x (2ax - x^2);$$

$$\int_x (2ax^2 - x^3) = \tfrac{2}{3} ax^3 - \tfrac{1}{4} x^4 + C;$$

$$(\int_{x=2a} - \int_{x=0}) (2ax^2 - x^3) = \tfrac{4}{3} . a^4;$$

$\therefore \int_x \int_y xy$, between the proper limits, $= \frac{2}{3} a^4$.

Therefore $X = \dfrac{5}{4} a, \quad Y = \dfrac{4}{3} . \dfrac{a}{\pi}$.

121. To find the center of pressure of the sector AOB (fig. 66.) having its center O in the surface of the fluid, and the radius OA perpendicular to the surface.

Let the plane of the sector meet the surface of the fluid in Oy. Draw the radii OR, OS; and with the centre O describe the arcs PP', QQ'. Let the density of the fluid $= \rho$, $OA = a$, $AOB = \alpha$, $OP = r$, $PQ' = \delta r$, $AOR = \theta$, $ROS = \delta\theta$.

Then d_r (press. on $PP'O$) δr = press. on PQ

$= g\rho r^2 \cos\theta . \delta\theta . \delta r$ ultimately;

$\therefore$ press. on $PP'O = g\rho \cos\theta . \delta\theta . \int_r r^2 = g\rho \cos\theta . \delta\theta \dfrac{r^3}{3} + C$;

$$\text{and press. on } ROS = g\rho \cos\theta \,.\, \delta\theta \left(\int_{r=a} - \int_{r=0}\right) r^2$$

$$= g\rho \cos\theta \,.\, \delta\theta \,.\, \frac{a^3}{3};$$

$$d_\theta \text{ (press. on } AOR) \,\delta\theta = \text{press. on } ROS$$

$$= \tfrac{1}{3}\, g\rho a^3 \cos\theta \,.\, \delta\theta \text{ ultimately};$$

$$\therefore \text{ press. on } AOR = \tfrac{1}{3}\, g\rho a^3 \int_\theta \cos\theta = \tfrac{1}{3}\, g\rho a^3 \sin\theta + C;$$

$$\text{and press. on } AOB = \tfrac{1}{3}\, g\rho a^3 \left(\int_{\theta=\alpha} - \int_{\theta=0}\right) \cos\theta = \tfrac{1}{3}\, g\rho a^3 \sin\alpha.$$

$$d_r \text{ (mom. press. on } PP'O \text{ round } Oy) \,\delta r$$

$$= \text{mom. press. on } PQ \text{ round } Oy$$

$$= g\rho r^3 (\cos\theta)^2 \,.\, \delta r \,.\, \delta\theta \text{ ultimately};$$

$$\therefore \text{ mom. press. on } PP'O \text{ round } Oy$$

$$= g\rho\, (\cos\theta)^2 \,.\, \delta\theta \int_r r^3 = g\rho\, (\cos\theta)^2 \,.\, \delta\theta\, \frac{r^4}{4} + C;$$

$$\text{and mom. press. on } ROS \text{ round } Oy$$

$$= g\rho\, (\cos\theta)^2 \,.\, \delta\theta \left(\int_{r=a} - \int_{r=0}\right) r^3,$$

$$= g\rho\, (\cos\theta)^2 \,.\, \delta\theta \,.\, \frac{a^4}{4};$$

$$d_\theta \text{ (mom. press. on } AOR \text{ round } Oy) \,.\, \delta\theta$$

$$= \text{mom. press. on } ROS \text{ round } Oy$$

$$= \tfrac{1}{4}\, g\rho a^4 (\cos\theta)^2 \,.\, \delta\theta \text{ ultimately};$$

$$\therefore \text{ mom. press. on } AOR \text{ round } Oy = \tfrac{1}{4}\, g\rho a^4 \int_\theta (\cos\theta)^2$$

$$= \tfrac{1}{16}\, g\rho a^4 (2\theta + \sin\theta) + C;$$

and mom. press. on AOB round Oy

$$= \tfrac{1}{16} g\rho a^4 \left(\textstyle\int_{\theta=\alpha} - \int_{\theta=0}\right) (\cos\theta)^2 = \tfrac{1}{16} g\rho a^4 (2\alpha + \sin\alpha).$$

d_r (mom. press. on $PP'O$ round OA) . δr

$=$ mom. press. on PQ round OA

$= g\rho r^3 \sin\theta \cos\theta \,.\, \delta r \,.\, \delta\theta$ ultimately;

$\therefore$ mom. press. on $PP'O$ round $OA = g\rho \sin\theta\cos\theta \,.\, \delta\theta \int_r r^3$

$$= g\rho \sin\theta\cos\theta \frac{r^4}{4} + C;$$

and mom. press. on ROS round OA

$$= g\rho \sin\theta\cos\theta \,.\, \delta\theta \left(\textstyle\int_{r=a} - \int_{r=0}\right) r^3 = g\rho \sin\theta \,.\, \cos\theta \,.\, \delta\theta \frac{a^4}{4};$$

d_θ (mom. press. on AOR round OA) δr

$=$ mom. press. on ROS round OA

$= \frac{1}{4} g\rho a^4 \sin\theta \cos\theta \,.\, \delta\theta$ ultimately;

$\therefore$ mom. press. on AOR round $OA = \frac{1}{4} g\rho a^4 \int_\theta \sin\theta\cos\theta$

$$= \tfrac{1}{8} g\rho a^4 (\sin\theta)^2 + C;$$

and mom. press. on AOB round OA

$$= \tfrac{1}{4} g\rho a^4 \left(\textstyle\int_{\theta=\alpha} - \int_{\theta=0}\right) \sin\theta\cos\theta = \tfrac{1}{8} g\rho a^4 (\sin\alpha)^2.$$

$$\therefore\ X \,.\, \tfrac{2}{3} g\rho a^3 \sin\alpha = \tfrac{1}{16} g\rho a^4 (2\alpha + \sin\alpha),$$

$$Y \,.\, \tfrac{1}{3} g\rho a^3 \sin\alpha = \tfrac{1}{8} g\rho a^4 (\sin\alpha)^2;$$

$$\therefore\ X = \frac{3}{16} a \left(2 \frac{\alpha}{\sin\alpha} + 1\right), \quad Y = \frac{3}{8} a \sin\alpha.$$

122. A hemispherical bell is placed with its mouth downwards on a horizontal plane, and water is poured into the bell through a hole in its vertex; to find how high the water will rise without lifting the bell.

Let BAB (fig. 67.) be a section of the bell made by a plane through its axis AC, BCB a section of the horizontal plane, PHP' a section of the surface of the water. Draw BQ parallel to AC meeting HP in Q; and let ρ be the density of the water; W the weight of the bell. The pressure of the water on the interior of the bell, estimated vertically upwards, is equal to the weight of the superincumbent column of fluid, or the weight of a quantity of fluid of the same bulk as the solid generated by the revolution of BPQ round $AC = \frac{1}{3}\pi g\rho HC^3$; and when $\frac{\pi}{3} g\rho HC^3 = W$, the weight of the bell is sustained by the pressure of the water.

123. A hollow sphere just filled with fluid, is divided into two parts by a vertical plane through its centre; the two hemispheres are held together by ligaments at their highest and lowest points; to find the tensions of the ligaments.

Let the circle APQ (fig. 68.) be the section of the sphere made by the vertical plane; P, Q, the highest and lowest points in the circle APQ, C its centre, K its centre of pressure. Then, (Art. 19.) the pressure on each hemisphere resolved in a direction perpendicular to APQ, is equal to the pressure on the circle APQ; and it acts in a line passing through K. Hence if P, Q, be the tensions of the ligaments at P, Q respectively, $P + Q =$ pressure on APQ; and $P \,.\, PQ =$ (pressure on APQ) $. \, KQ$.

If the radius of the sphere $= a$, and the density of the fluid $= \rho$, the pressure on $APQ = g\rho\pi a^3$,

$$\text{and (Art. 20.) } PK = \tfrac{5}{4}a; \quad \therefore KQ = \tfrac{3}{4}a;$$

$$\therefore P + Q = g\rho\pi a^3, \quad P \,.\, 2a = g\rho\pi a^3 \,.\, \tfrac{5}{4}a,$$

$$\therefore P = \tfrac{3}{8} g\rho\pi a^3, \quad Q = \tfrac{5}{8} g\rho\pi a^3.$$

124. A rod AB (fig. 69.) of uniform thickness, suspended by a string EL, rests with one end immersed in a fluid; to find AP the portion of the rod immersed, and the tension of the string by which it is suspended.

Let κ be the area of a section of the rod, W its weight, G its centre of gravity, ρ the density of the fluid. Bisect AP in F; and through F, G draw FM, GN, vertical. The resultant of the pressure of the fluid on the rod = weight of the fluid displaced $= g\rho K . AP$; and it acts in the line FM; the other forces are W acting in GN, and T in EL. Therefore, (Art. 22.)

$$T + g\rho\kappa . AP = W; \quad g\rho\kappa . AP . FE = W . GE,$$

$$\therefore \; AP^2 - 2AE . AP + 2\frac{W}{g\rho\kappa} = 0,$$

from this equation AP and therefore T may be found.

125. A ship sailing out of the sea into a river, sinks through the space b; on throwing overboard a weight P the ship rises through the space c; to find the weight of the ship.

Let ρ, σ be the densities of fresh and salt water respectively, A the area of the plane of floatation of the ship, W its weight, V the volume of the salt water displaced by the ship; then, (Art. 22. Cor. 2.) $W = g\sigma V$, and the volume of the fresh water displaced at first $= V + bA$, $\therefore W = g\rho\,(V + bA)$; and the volume of the fresh water displaced after the weight P is thrown overboard $= V + (b - c)\,A$, $\therefore W - P = g\rho\,\{V + (b - c)\,A\}$.

Eliminating A, V, we obtain $\left(1 - \dfrac{\rho}{\sigma}\right) W = \dfrac{b}{c} P$.

126. A triangular prism floats with its axis horizontal, and one edge immersed; to find its positions of equilibrium.

Let RSD, AB (fig. 70.) be sections of the prism and of the plane of floatation, made by a plane perpendicular to the

axis of the prism, passing through G its centre of gravity. Let W be the weight of the prism, h the length of its axis, ρ the density of the fluid. Draw GE perpendicular to RD, GF perpendicular to SD. Take $PD = \frac{2}{3} AD$, $QD = \frac{2}{3} BD$; and bisect PQ in H. Then H is the centre of gravity of the fluid displaced; Therefore (Art. 22. Cor. 2.) GH is perpendicular to AB or PQ; and $\frac{1}{2} g\rho h . AD . BD . \sin D = W$.

$$PG^2 = DG^2 + PD^2 - 2ED . PD,$$

$$QG^2 = DG^2 + QD^2 - 2FD . QD;$$

and $PG = QG$, for $PH = QH$, and GH is perpendicular to PQ;

$$\therefore PD^2 - QD^2 - 2ED . PD + 2FD . QD = 0;$$

$$\therefore AD^2 - BD^2 - 3ED . AD + 3FD . BD = 0;$$

$$\text{and } BD = \frac{2W}{g\rho h \sin D . AD};$$

$$\therefore AD^4 - 3ED . AD^3 + \frac{6FD . W}{g\rho h \sin D} AD - \frac{4W^2}{g^2\rho^2 h^2 (\sin D)^2} = 0.$$

The last term of this equation is negative, and therefore one root of it is negative; but the nature of the question excludes all negative values of AD and BD. Hence, there cannot be more than three positions of equilibrium as long as the same edge is immersed. All values of AD greater than RD, and of BD greater than SD are likewise inadmissible.

127. Two equal rods RD, SD, (fig. 71.) meeting each other at right angles, float with the angle D immersed; to find their positions of equilibrium.

Let G be the centre of gravity of the rods; P, Q, the middle points of the portions immersed; GR perpendicular to DR; GS perpendicular to DS; GH perpendicular to PQ; HN perpendicular to RD; $2c$ the sum of the lengths of the

immersed portions, when the weight of the fluid displaced is equal to the weight of the rods; $RD = e$; $PD = a$; $QD = b$. Then, $a + b = c$; and the centre of gravity of the fluid displaced must be in GH, it must also be in PQ, therefore H is the centre of gravity of the fluid displaced;

$$\therefore \frac{b}{a} = \frac{PH}{QH} = \frac{PD}{DN}, \quad \therefore \frac{a+b}{a} = \frac{PD}{ND}, \quad \therefore ND = \frac{a^2}{c},$$

The equation to PQ referred to the axes DR, DS, is $\frac{y}{b} + \frac{x}{a} = 1$; the co-ordinates of G are e, e', therefore the equation to GH is $b(y - e) = a(x - e)$; and H is the intersection of GH and PQ,

$$\therefore (b^2 + a^2) DN = ab^2 + (a^2 - ab)e, \quad \therefore (b^2 + a^2)a = cb^2 + (a - b)ec,$$

$$\therefore \{(c - a)^2 + a^2\} a = c(c - a)^2 + (2a - c)ec,$$

$$\therefore 2a^3 - 3ca^2 + (3c - 2e)ac - (c - e)c^2 = 0,$$

one root of this equation is $\frac{1}{2}c$, the other two are

$$\tfrac{1}{2}\left\{c + \sqrt{\left(4\frac{e}{c} - 3\right)}\right\}, \quad \tfrac{1}{2}\left\{c - \sqrt{\left(4\frac{e}{c} - 3\right)}\right\}.$$

128. To find (M) the metacentre of the prism RDS (fig. 70), the prism being inclined in the plane RDS.

The moment of inertia of the plane of floatation round an axis through its centre of gravity, perpendicular to RDS

$$= \tfrac{1}{3} \cdot \left(\frac{AB}{2}\right)^2 . AB . h;$$

and the volume of the fluid displaced $= \frac{1}{2} h . AD . DB . \sin D$;

$$\therefore \text{(Art. 26.)}\ \tfrac{1}{2} . h . AD . DB . \sin D . HM = \tfrac{1}{12} AB^3 . h,$$

$$\therefore HM = \frac{1}{6} \frac{AB^3}{AD . DB \sin D}.$$

129. To find the metacentre of a cone floating with its axis vertical.

Let DC (fig. 72.) be the axis of the cone, meeting the plane of floatation in C, CA the radius of the plane of floatation, H the centre of gravity of the fluid displaced. Then $DH = \frac{3}{4} DC$; the moment of inertia of the plane of floatation round a horizontal axis through its centre of gravity $C = \frac{\pi}{4} AC^4$;

and the volume of the fluid displaced $= \frac{\pi}{3} AC^2 . DC$;

$$\therefore \frac{\pi}{3} AC^2 . DC . HM = \frac{\pi}{4} AC^4,$$

$$\therefore HM = \frac{3}{4} \frac{AC^2}{DC}; \quad DM = \frac{3}{4} \frac{DC^2 + AC^2}{DC} = \frac{3}{4} \frac{AD^2}{DC}.$$

130. A conical vessel partly filled with fluid, floats in the same fluid with its axis vertical; to find whether the equilibrium of the vessel is stable or unstable.

Let DM (fig. 73.) be the axis of the cone making a very small angle with the vertical; C, c the points in which it cuts the plane of floatation and the surface of the fluid within; H, h the centres of gravity of the fluid displaced, and of the fluid contained in the cone, when the axis of the cone was vertical; M, m the points in which verticals through the centres of gravity of the fluid displaced, and of the fluid within ultimately intersect DM; G the centre of gravity of the cone. The weight of the fluid displaced, the weight of the cone, and the weight of the fluid contained in the cone, act in parallel lines through M, G, m, respectively. And the pressures of the exterior and interior fluids will tend to diminish or increase the inclination of DC according as (weight of fluid bda); mG is greater or less than (weight of fluid displaced); MG.

$$MD = \frac{3}{4} \frac{AD^2}{CD}, \quad mD = \frac{3}{4} \frac{aD^2}{cD},$$

weight of fluid bda : weight of fluid displaced $= cD^3 : CD^3$; therefore the equilibrium of the cone will be stable or unstable according as $cD^3\left(GD - \frac{3}{4}\frac{aD^2}{cD}\right)$ is greater or less than

$$CD^3\left(\frac{3}{4}\frac{AD^2}{CD} - GD\right).$$

131. An open vessel containing fluid, is made to revolve round a vertical axis with the angular velocity a; to find the form of the surface of the fluid.

Let the axis of revolution be made the axis of z; and let z be measured downwards. The forces on the fluid at any point P whose co-ordinates are x, y, z, are, g acting downwards, and $a^2\sqrt{(x^2+y^2)}$ acting in the direction of a perpendicular from P on the axis of revolution; this force may be resolved into $a^2\sqrt{(x^2+y^2)}\dfrac{x}{\sqrt{(x^2+y^2)}}$, or a^2x, in a direction parallel to the axis of x, and a^2y, in a direction parallel to the axis of y. We have then,

$$X = a^2x,\ Y = a^2y,\ Z = g;\ \therefore\ d_xp = \rho a^2x,\ d_yp = \rho a^2y,\ d_zp = \rho g;$$

$$\therefore\ p = \rho\left\{\tfrac{1}{2}a^2(x^2+y^2) + gz\right\} + C.$$

Let the surface of the fluid cut the axis of z at the depth c below the origin; and let Π be the pressure of the atmosphere.

Then, $\Pi = \rho gc + C$;

$$\therefore\ p - \Pi = \rho\left\{\tfrac{1}{2}a^2(x^2+y^2) + g(z-c)\right\}.$$

At any point in the surface of the fluid $p = \Pi$, therefore the equation to the surface of the fluid is

$$0 = a^2(x^2+y^2) + 2g(z-c);$$

the equation to a paraboloid generated by the revolution of a parabola whose latus rectum $= \dfrac{2g}{a^2}$.

132. A hollow parallelopiped OF (fig. 74.) just filled with fluid, revolves round the edge OC, which is vertical, with the angular velocity α; to find the pressure on the side BF, and the centre of pressure of the side BF.

Let $OA = a$, $OB = b$, $OC = c$; OA, OB, OC, the axes of x, y, z, respectively; p the pressure at the point (x, y, z).

Then,

$$p = \rho \left\{\tfrac{1}{2}\alpha^2 (x^2 + y^2) + gz\right\} + C;$$

$$\text{at } O \;\; p = 0, \;\; x = 0, \;\; y = 0, \;\; z = 0, \quad \therefore C = 0,$$

$$\therefore \; p = \rho \left\{\tfrac{1}{2}\alpha^2 (x^2 + y^2) + gz\right\}.$$

Draw HPR, KQS, parallel to BD; MP, NQ parallel to BE; and let $BM = x$, $BH = z$.

The pressure at $P = \rho \left\{\tfrac{1}{2}(x^2 + b^2) + gz\right\}$;

the pressure on $PQ = \rho \left\{\tfrac{1}{2}(x^2 + b^2) + gz\right\} MN \,.\, HK$ ult.

the pressure on $KP = \rho \left\{\tfrac{1}{2}(\tfrac{1}{3}x^3 + b^2 x) + gzx\right\} . \, HK$ ult.

the pressure on $KR = \rho \left\{\tfrac{1}{2}(\tfrac{1}{3}a^3 + b^2 a) + gza\right\} . \, HK$ ult.

the pressure on $BR = \rho \left\{\tfrac{1}{2}(\tfrac{1}{3}a^3 + b^2 a) z + \tfrac{1}{2}gz^2 a\right\}$;

the pressure on $BF = \rho \left\{\tfrac{1}{2}(\tfrac{1}{3}a^3 + b^2 a) c + \tfrac{1}{2}gc^2 a\right\}$.

The moment of the pressure on PQ round BE

$$= \rho \left\{\tfrac{1}{2}\alpha^2 (x^2 + b^2) + gz\right\} x \,.\, MN \,.\, HK \text{ ult.}$$

the moment of the pressure on KP

$$= \rho \left\{\tfrac{1}{2}\alpha^2 (\tfrac{1}{4}x^4 + \tfrac{1}{2}b^2 x^2) + \tfrac{1}{2}gzx^2\right\} . \, HK \text{ ult.}$$

the moment of the pressure on KR

$$=\rho\left\{\tfrac{1}{2}a^2\left(\tfrac{1}{4}a^4+\tfrac{1}{2}b^2a^2\right)+\tfrac{1}{2}gza^2\right\}.HK \text{ ult.}$$

the moment of the pressure on BR

$$=\rho\left\{\tfrac{1}{2}a^2\left(\tfrac{1}{4}a^4+\tfrac{1}{2}b^2a^2\right)z+\tfrac{1}{4}gz^2a^2\right\};$$

the moment of the pressure on BF round BE

$$=\rho\left\{\tfrac{1}{2}a^2\left(\tfrac{1}{4}a^4+\tfrac{1}{2}b^2a^2\right)c+\tfrac{1}{4}gc^2a^2\right\}.$$

The moment of the pressure on PQ round BD

$$=\rho\left\{\tfrac{1}{2}a^2\left(x^2+b^2\right)+gz\right\}z.MN.HK \text{ ult.}$$

the moment of the pressure on KP

$$=\rho\left\{\tfrac{1}{2}a^2\left(\tfrac{1}{3}x^3+b^2x\right)+gzx\right\}z.HK \text{ ult.}$$

the moment of the pressure on KR

$$=\rho\left\{\tfrac{1}{2}a^2\left(\tfrac{1}{3}a^3+b^2a\right)+gza\right\}z.HK \text{ ult.}$$

the moment of the pressure on BR

$$=\rho\left\{\tfrac{1}{4}a^2\left(\tfrac{1}{3}a^3+b^2a\right)z^2+\tfrac{1}{3}gz^3a\right\};$$

the moment of the pressure on BF round BD

$$=\rho\left\{\tfrac{1}{4}a^2\left(\tfrac{1}{3}a^3+b^2a\right)c^2+\tfrac{1}{3}gc^3a\right\}.$$

Hence if X, Z, be the co-ordinates of the centre of pressure of BF, referred to the axes BD, BE,

$$X\left\{\tfrac{1}{2}a^2\left(\tfrac{1}{3}a^2+b^2\right)+\tfrac{1}{2}gc\right\}=\tfrac{1}{2}a^2\left(\tfrac{1}{4}a^3+\tfrac{1}{2}b^2a\right)+\tfrac{1}{4}gca.$$

$$Z\left\{\tfrac{1}{2}a^2\left(\tfrac{1}{3}a^2+b^2\right)+\tfrac{1}{2}gc\right\}=\tfrac{1}{4}a^2\left(\tfrac{1}{3}a^2+b^2\right)c+\tfrac{1}{3}gc^2.$$

133. Two plates of glass meeting in the vertical Cy (fig. 75.), and making a very small angle ϵ with each other, are immersed in water; to find the figure of the water elevated between them by capillary attraction.

Let one of the plates meet the surface of the fluid between them in PQ, and the plane of undisturbed surface in Cx.

Draw PN parallel to Cy. Let $CN = x$, $PN = Y$,

the distance between the plates at $P = \epsilon x$;

Therefore (Art. 48.) $\epsilon xy = \dfrac{H}{g}$ nearly, the equation to a rectangular hyperbola of which Cx, Cy, are the asymptotes.

134. A wire, the area of a section of which $= \kappa$, can just sustain a weight W without breaking; to find the greatest pressure that can be applied to a fluid contained in a hollow cylinder of the same substance as the wire, without bursting it, a being the radius of the cylinder, and e its thickness.

Let ML (fig. 25.) be a portion of the cylinder; MK, HL perpendicular to its axis; MH, KL parallel to its axis; p the pressure of the fluid. The area of the section $MH = e \,.\, MH$, therefore it can sustain a tension $\dfrac{e \,.\, MH}{\kappa} W$; and (Art. 49.)

$$p \,.\, MH = \frac{e \,.\, MH}{\kappa a} W, \quad \therefore \; p = \frac{e}{\kappa a} W.$$

A pressure $\dfrac{2e}{\kappa a} W$ might be applied to a hollow sphere of the same radius and thickness without bursting it.

135. To find the time of emptying a vertical prism or cylinder through a small orifice in its base.

Let A be the area of the base of the prism, κ the area of the orifice, x the depth of the orifice below the surface of the

fluid at the end of the time t from the beginning of the motion. Then, (Art. 58.)

$$\sqrt{(2g)}\,\kappa d_x t = -\frac{A}{\sqrt{x}};$$

$$\therefore \sqrt{(2g)}\,\kappa t = C - 2A\sqrt{x};$$

if the depth of the orifice below the surface of the fluid was a when $t = 0$.

$$0 = C - 2A\sqrt{a};$$

$$\therefore \sqrt{(2g)}\kappa . t = 2A(\sqrt{a} - \sqrt{x});$$

and the whole time of emptying

$$= \frac{A}{\kappa}\sqrt{\left(\frac{2a}{g}\right)}.$$

136. To find the time of emptying a hollow sphere through a small orifice in its vertex.

Let a be the radius of the sphere, κ the area of the orifice, x the depth of the orifice below the surface of the fluid at the end of the time t from the beginning of the motion.

The area of the surface of the fluid at the end of the time t

$$= \pi(2ax - x^2),$$

$$\therefore \sqrt{(2g)}\kappa d_x t = -\frac{\pi(2ax - x^2)}{\sqrt{x}} = -\pi(2ax^{\frac{1}{2}} - x^{\frac{3}{2}});$$

$$\therefore \sqrt{(2g)}\kappa t = C - \pi\left(\frac{4}{3}ax^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}\right);$$

when $t = 0$, $x = 2a$,

$$\therefore 0 = C - \frac{16}{15}\sqrt{2}\pi a^{\frac{5}{2}};$$

$$\therefore \sqrt{(g)}\kappa t = \tfrac{16}{15}\pi a^{\frac{5}{2}} - \tfrac{4}{3}\pi a x^{\frac{3}{2}} + \tfrac{2}{5}\pi x^{\frac{5}{2}};$$

and the time of emptying the whole sphere $= \dfrac{16\pi a^{\frac{5}{2}}}{15\kappa\sqrt{(g)}}$.

137. To determine the motion of a fluid oscillating in an inverted syphon PDB (fig. 76.) of uniform bore.

Let P, Q be the extremities of the column of fluid at the end of the time t from the beginning of the motion; A, B, the extremities of the column of fluid when at rest; α, β the angles between AP, BQ and the vertical MN; $AP = s$, κ the area of a section of the tube. The moving force on the fluid $= g\rho\kappa . MN$

$$= g\rho\kappa(AP\cos\alpha + BQ\cos\beta) = g\rho\,(\cos\alpha + \cos\beta)AP.$$

The mass of the fluid $= \rho\kappa . ADB$, and, since the bore of the tube is uniform, every part of the fluid moves with the same velocity, therefore the effective accelerating force at any point tending to make the fluid return to its position of equilibrium

$$= g(\cos\alpha + \cos\beta)\frac{AP}{ADB}; \quad \therefore d_t^2 s + g(\cos\alpha + \cos\beta)\frac{s}{ADB} = 0;$$

and the time of an oscillation $= \pi\sqrt{\dfrac{ADB}{g(\cos\alpha + \cos\beta)}}$.

138. A weight is raised by a rope wound round the axle of an undershot wheel; to find the velocity of the wheel.

Let K be the area of each float-board; u the velocity of the wheel; v the velocity of the stream; a the radius of the wheel; b the radius of the axle; W the weight. The relative velocity of the stream is $v - u$, and, therefore, the force with which it impels the wheel $= \frac{1}{2}\rho K(v-u)^2$. And when the velocity of the wheel is uniform, this force is in equilibrium with the weight W, $\therefore bW = a\frac{1}{2}\rho K(v-u)^2$.

The work performed by a water wheel is measured by the product of the weight lifted multiplied by the velocity of the weight. (W's velocity) $= W\dfrac{b}{a}u = \frac{1}{2}\rho\,\mathrm{K}(v-u)^2u$; this is a maximum when $3u = v$. The weight lifted in this case

$$= \frac{2}{g}\,\frac{a}{b}\rho\,\mathrm{K}v^2.$$

The wheel would be kept at rest by a weight $\dfrac{a}{b}\rho\,\mathrm{K}v^2$, therefore the work performed by the wheel is a maximum when the weight lifted is $\frac{4}{9}$ of the weight that would keep the wheel at rest.

139. To find the position of the rudder of a ship, when the effect of the rudder in turning the ship is a maximum.

Let AP (fig. 33.) be the keel of the ship, PE perpendicular to the rudder. The resolved part of the resistance on the rudder estimated in a direction perpendicular to AP,

$$\propto (\cos APE)^2 . \sin APE. \quad \text{(Art. 69. Cor. 2.)};$$

And this is a maximum when

$$0 = (\cos APE)^3 - 2(\sin APE)^2 . \cos APE,$$

$$\text{or } \sin APE = \tfrac{1}{3}\sqrt{3}.$$

The theory of resistances (Art. 68.) is the same as that given in the Note, Page 188, of Mr. Moseley's Hydrostatics. The correctness of the application of the theorem (Art. 53.) in this case, appears doubtful.

140. Example of the comparison of the specific gravities of two fluids.

A glass flask being filled with mercury at 20,6, the mercury appeared to weigh 1340,893 grammes; when filled with water at 20,5, the water appeared to weigh 98,7185 grammes; the weight of the air contained in the flask = 0.1186 grammes; therefore

the true weight of the water $= 98,8371$; and the true weight of the mercury $= 1341,0116$. The apparent expansion of mercury in glass, between 0° and 100°, $= 0.0154$, therefore the true weight of the mercury contained in the flask at $20^\circ,5$ $= 1341.0116 + (.0000154)(1341) = 1341,0323$; $\therefore$ ($S.G.$mercury) : ($S.G.$water), at $20^\circ,5 = .(1341,0323) : (98,8371) = 13,5681$.

Between 0^0 and $20^0,5$ the expansion of mercury $= 0.00369$, and the expansion of water $= 0.0017$; therefore ($S.G.$mercury) : ($S.G.$water), at $0^0 = 13,5681\ (1 + .00369 - .0017) = 13,5952$.

141. Daniell's barometric formula.

Let u, v be the altitudes of the columns of mercury supported by the pressure of the vapour contained in the air at the lower and upper stations respectively: then, retaining the notation of (Arts. 35, 36.) the mean ratio of the pressure to the density at 0^0 will be

$$\mu\left\{1 + \tfrac{3}{8}\,\tfrac{1}{2}\left(\frac{u}{h} + \frac{v}{k}\right)\right\}, \text{ (Art. 66.) Therefore}$$

$$x = \log_e 10\,\frac{\mu}{g'}\left\{1 + \frac{3}{16}\left(\frac{u}{h} + \frac{v}{k}\right)\right\}\left(1 + \frac{x}{r}\right)\left\{1 + \tfrac{1}{2}\,\text{E}(\text{S} + \text{T})\right\},$$

$$\times\left\{\log_{10} h - \log_{10} k - \log_{10} e.e(s - t) + \log_{10} e 2\,\frac{x}{r}\right\}.$$

$$\frac{\mu}{g} = (10467)\,(29,9218) \text{ inches} = 26099 \text{ feet (Art. 31.)}$$

$$\log_e 10.\frac{\mu}{g} = 60095 \text{ feet.} \quad \text{E} = 0.00375.$$

$$\therefore x = \left\{60095 + (112,7)(\text{S} + \text{T}) + [1127 + 21(\text{S} + \text{T})]\left(\frac{u}{h} + \frac{v}{k}\right)\right.$$

$$\left. + 155.\cos 2\lambda + (.0029)x\right\}$$

$$\times\left\{\log_{10} h - \log_{10} k - (0.000078)(s - t) + (0.0000000416)x\right\}.$$

When x is not very large, $x =$

$$\left\{60282 + 113(\mathrm{S} + \mathrm{T}) + [1130 + 21\,(\mathrm{S} + \mathrm{T})]\left(\frac{u}{h} + \frac{v}{k}\right) + 155.\cos 2\lambda\right\}$$

$$\times \left\{\log_{10} h - \log_{10} k - (0.000078)\,(s - t)\right\}.$$

142. If T be the temperature of steam, u its pressure expressed in inches of mercury at 0^0, it is found that as long as T is not much greater than 100,

$$\log_{10} u = 1{,}4759877 - (0{,}01537278)\,(100 - \mathrm{T}) - (0{,}00006732)\,(100 - \mathrm{T}^2)$$

$$+ (0{,}00000003374)\,(100 - \mathrm{T})^3.$$

And for temperatures between 100^0 and 230^0,

$$\log_{10} u = 1{,}4759877 + 5.\log_{10}\left\{1 + (0.007153)\,(\mathrm{T} - 100)\right\}.$$

143. According to Dr. Young, the expansion of water is expressed very nearly by the formula

$$\mathrm{E} = (0.0000063475)\,(\mathrm{T} - 3{,}9)^2 - (0.000000013865)\,(\mathrm{T} - 3{,}9)^3.$$

144. Ratios of the specific gravities of different substances to that of water at 60° *F* or $15{,}5^\circ C$.

PLATINA	20,98	Iceland Spar	2,718
Gold	19,257	Quartz	2,6
Mercury	13,568	Plate Glass	2,4
Lead	11,352	Sulphur	2,
Silver	10,474	Common Salt	1,918
Bismuth	9,07	Ivory	1,917
Copper	8,895	Plumbago	1,9
Iron	7,788	Amber	1,078
Tin	7,291	Sea Water	1,027
Zinc	6,861	Wax	0,96
Corundum	4,	Ice	0,95
Diamond	3,5	Olive Oil	0,9153
Flint Glass	3,33	Alcohol	0,796
Marble	2,716	Naphtha	0,758

Ratios of the densities of gases to that of water, the gases being at 0°, under the pressure of 29,922 inches of mercury at 0°, in latitude 45, and the water at 4°.

Atmospheric Air	0,0012991
Chlorine	0,003211
Carbonic Acid	0,0019805
Oxygen	0,0014323
Nitrogen	0,0012675
Vapour of Water	0,00081
Ammonia	0,0007752
Hydrogen	0,0000894

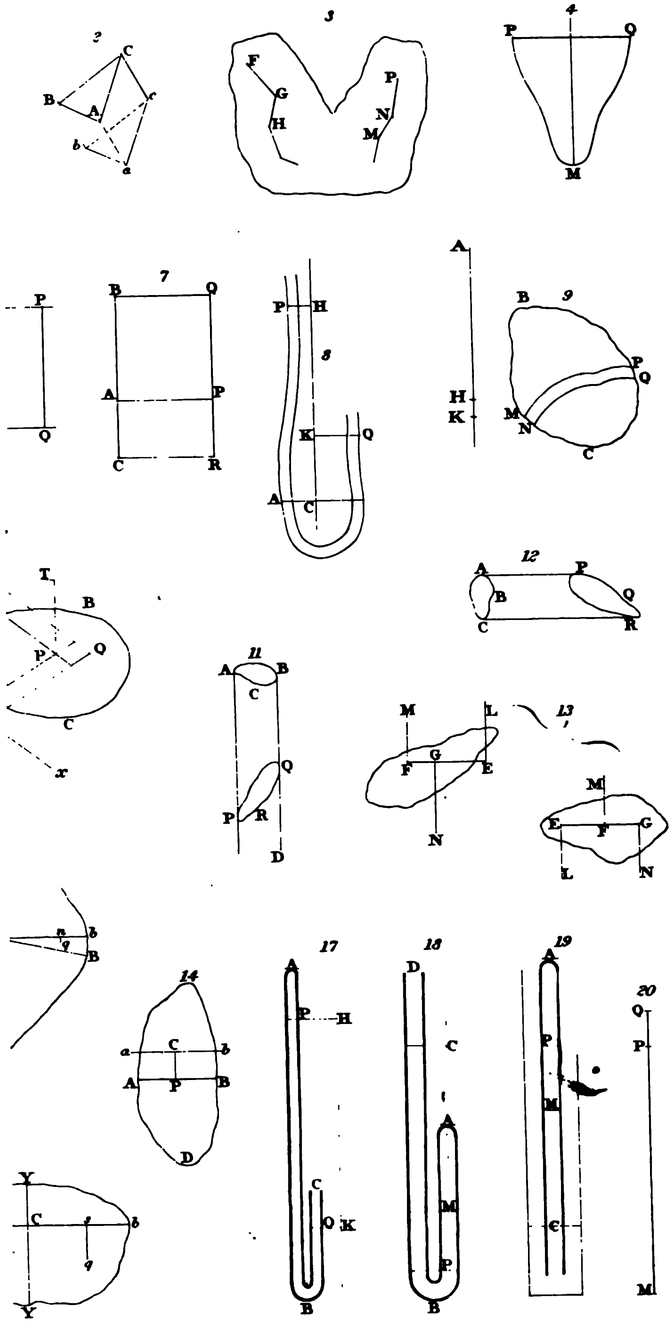
The weight of a cubic inch of water at 4° = 253,032 grains.

The expansion of gases between 0° and 100°, under a constant pressure, is equal to that of atmospheric air, or 0.375 of the volume of the gas at 0°. And their pressure is, within certain limits, inversely proportional to the space they occupy.

ERRATA.

Page	Line	
16	3	From the bottom, *for* C *read* P.
—	2	From the bottom, *for* C *read* P.
17	4	From the bottom, *add* "when the variation of the density is small".
25	3	From the bottom, *for* $d_r \phi r - \phi r$ *read* $d_r \phi r - \frac{1}{r} \phi r$.
54	22	From the bottom, *for* A *read* H.
55	11	From the ~~bottom~~ top, *for* A *read* B.
—	4	From the bottom, *for* a *read* h.

Plate 1.



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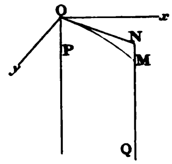

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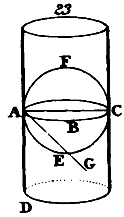

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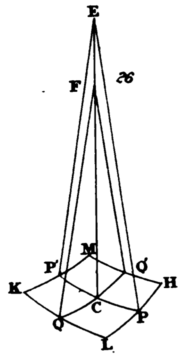

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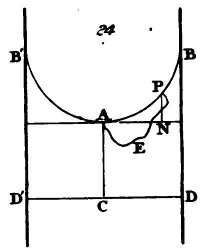

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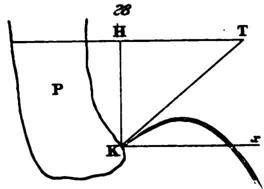

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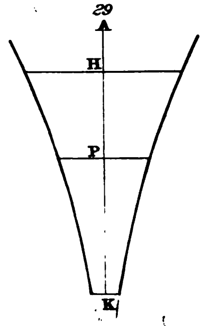

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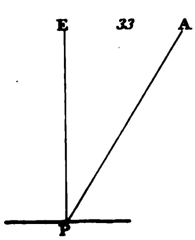

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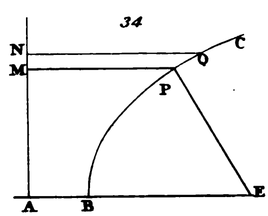

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Plate 3.

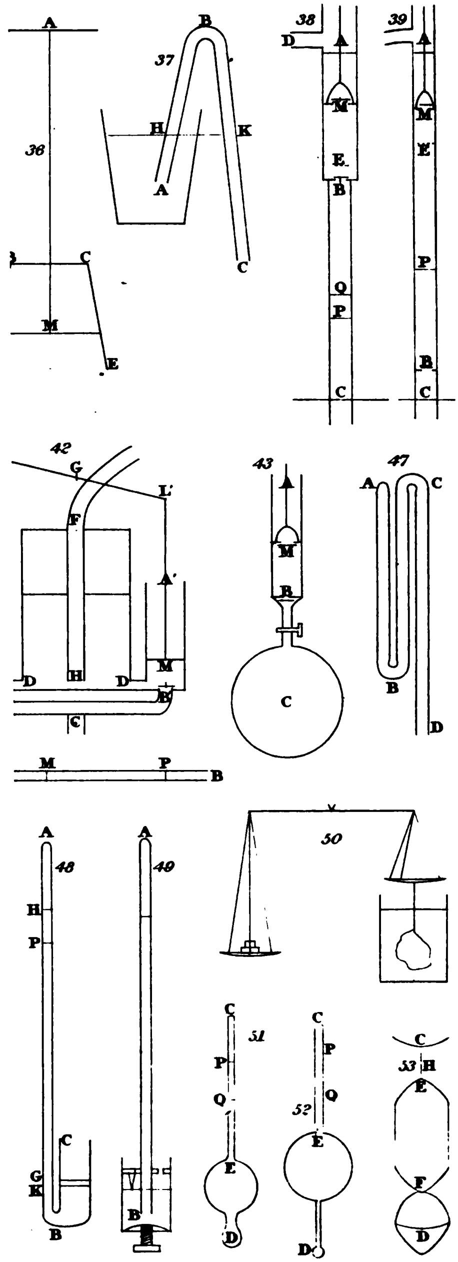

Plate 4.

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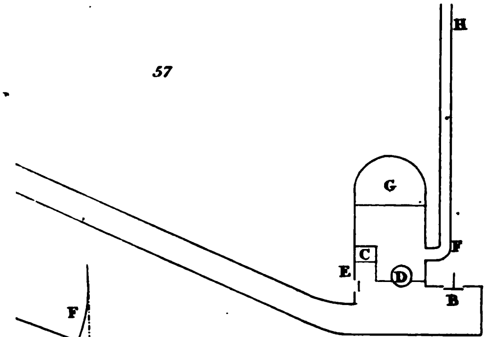

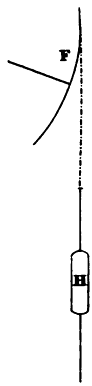

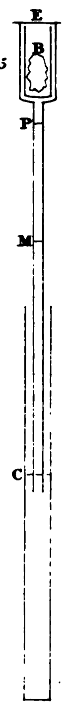

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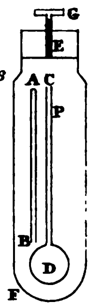

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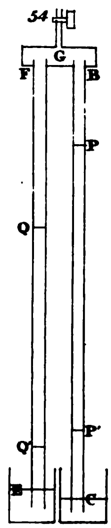

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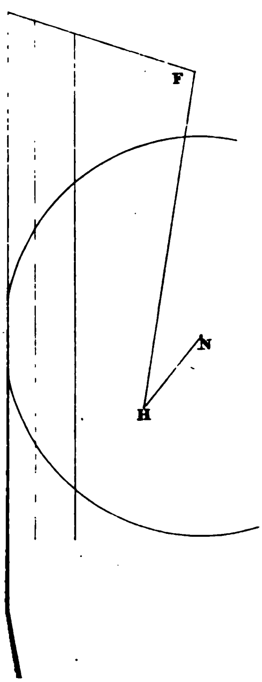

Engraved by Josiah Neele 352 Strand.

Plate 5.